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The extremal process of two-speed branching Brownian motion*

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Abstract

We construct and describe the extremal process for variable speed branching Brownian motion, studied recently by Fang and Zeitouni, for the case of piecewise constant speeds; in fact for simplicity we concentrate on the case when the speed is σ_1 for $s \leq bt$ and σ_2 when $bt \leq s \leq t$. In the case $\sigma_1 > \sigma_2$, the process is the concatenation of two BBM extremal processes, as expected. In the case $\sigma_1 < \sigma_2$, a new family of cluster point processes arises, that are similar, but distinctively different from the BBM process. Our proofs follow the strategy of Arguin, Bovier, and Kistler.

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1 Introduction

A standard branching Brownian motion (BBM) is a continuous-time Markov branching process that is constructed as follows: start with a single particle which performs a standard Brownian motion $x(t)$ with $x(0) = 0$ and continues for a standard exponentially distributed holding time T, independent of x. At time T, the particle splits independently of x and T into k offspring with probability p_k, where $\sum_{i=1}^{\infty} p_k = 1$, $\sum_{k=1}^{\infty} kp_k = 2$ and $K = \sum_{k=1}^{\infty} k(k-1)p_k < \infty$. These particles continue along independent Brownian paths starting from $x(T)$ and are subject to the same splitting rule. And so on.

Branching Brownian motion has received a lot of attention over the last decades, with a strong focus on the properties of extremal particles. We mention the seminal contributions of McKean [18], Bramson, Lalley and Sellke, and Chauvin and Rouault [7, 6, 15, 8] on the connection to the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP)

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equation and on the distribution of the rescaled maximum. In recent years, their has been a revival of interest in BBM with numerous contributions, including the construction of the full extremal process [3, 1]. For a review of these developments see, e.g., the recent survey by Guéré [13].

BBM can be seen as a Gaussian process with covariances depending on an ultrametric distance, in this case the ultrametric associated to the genealogical structure of an underlying Galton-Watson process. In that respect it is closely related to another class of Gaussian processes, the Generalised Random Energy Models (GREM) introduced by Derrida and Gardner [12]. While in BBM the covariance of the process is a linear function of the ultrametric distance, in the GREM one considers more general functions. One of the reasons that makes BBM interesting in this context is the fact that the linear function appears as a borderline where the correlation starts to modify the behaviour of extremes [4, 5].

In the context of BBM, different covariances can be achieved by varying the speed (i.e. diffusivity) of the Brownian motions as a function of time (see also [5]). This model was introduced by Derrida and Spohn [9] and has recently been investigated by Fang and Zeitouni [11, 10], see also [16, 17]. The entire family of models obtained as time changes of BBM is a splendid test ground to further develop the theory of extremes of correlated random variables. Understanding fully the possible extremal processes that arise in this class should also provide us with candidate processes for even wider classes of random structures.

1.1 Results

In [11], Fang and Zeitouni showed that in the case when the covariance is a piecewise linear function, the maximum of BBM is tight and behaves as expected from the analogous GREM. In this paper we refine and extend their analysis: we obtain the precise law of the maximum, and we give the full characterisation of the extremal process.

For simplicity we consider the following variable speed BBM. Fix a time t. Then we consider the BBM model where at time s, all particles move independently as Brownian motions with variance

$$\sigma^2(s) = \begin{cases} \sigma_1^2 & 0 \le s < bt \\ \sigma_2^2 & t \le s \le t \end{cases}, \quad 0 < b \le 1. \tag{1.1}$$

We normalise the total variance by assuming

$$\sigma_1^2 b + \sigma_2^2 (1-b) = 1. \tag{1.2}$$

Note that in the case $b = 1$, $\sigma_2 = \infty$ is allowed.

We denote by $n(s)$ the number of particles at time s and by $\{x_i(s); 1 \le i \le n(s)\}$ the positions of the particles at time s.

Remark 1.1. *Strictly speaking, we are not talking about a single stochastic process, but about a family $\{x_k(s), k \le n(s)\}_{s\le t}^{t\in\mathbb{R}_+}$ of processes with finite time horizon, indexed by that horizon, t.*

In this case, Fang and Zeitouni [10] showed that

$$\max_{k\le n(t)} x_k(t) = \begin{cases} \sqrt{2}t - \frac{1}{2\sqrt{2}} \log t + O(1), & \text{if } \sigma_1 < \sigma_2, \\ \sqrt{2}t(b\sigma_1 + (1-b)\sigma_2) - \frac{3}{2\sqrt{2}}(\sigma_1+\sigma_2)\log t + O(1), & \text{if } \sigma_1 > \sigma_2. \end{cases} \tag{1.3}$$

The second case has a simple interpretation: the maximum is achieved by adding to the maxima of BBM at time bt the maxima of their offspring at time $(1-b)t$ later. The first case looks simpler even, but is far more interesting. The order of the maximum is that of the REM, a fact to be expected by the corresponding results in the GREM (see [12, 4]). But what is the law of the rescaled maximum and what is the corresponding extremal process? The purpose of this paper is primarily to answer this question.

For standard BBM, $\bar{x}(t)$, (i.e. $\sigma_1 = \sigma_2$), Bramson [7] and Lalley and Sellke [15] show that

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{k\leq n(t)} \bar{x}_k(t) - m(t) \leq y\right) = \omega(x) = \mathbb{E}e^{-CZe^{-\sqrt{2}y}}, \tag{1.4}$$

where $m(t) \equiv \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$, Z is a random variable, the limit of the so called *derivative martingale*, and C is a constant.

In [3] (see also [1] for a different proof) it was shown that the extremal process,

$$\lim_{t\uparrow\infty} \widetilde{\mathcal{E}}_t \equiv \lim_{t\uparrow\infty} \sum_{k=1}^{n(t)} \delta_{\bar{x}_k(t)-m(t)} = \widetilde{\mathcal{E}}, \tag{1.5}$$

exists in law, and $\widetilde{\mathcal{E}}$ is of the form

$$\widetilde{\mathcal{E}} = \sum_{k,j} \delta_{\eta_k+\Delta_j^{(k)}}, \tag{1.6}$$

where η_k is the k-th atom of a mixture of Poisson point process with intensity measure $CZe^{-\sqrt{2}y}dy$, with C and Z as before, and $\Delta_i^{(k)}$ are the atoms of independent and identically distributed point processes $\Delta^{(k)}$, which are the limits in law of

$$\sum_{j\leq n(t)} \delta_{\tilde{x}_i(t)-\max_{j\leq n(t)} \tilde{x}_j(t)}, \tag{1.7}$$

where $\tilde{x}(t)$ is BBM conditioned on $\max_{j\leq n(t)} \tilde{x}_j(t) \geq \sqrt{2}t$.

The main result of the present paper is similar but different.

Theorem 1.2. *Let $x_k(t)$ be branching Brownian motion with variable speed $\sigma^2(s)$ as given in (1.1). Assume that $\sigma_1 < \sigma_2$. Then*

(i)

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{k\leq n(t)} x_k(t) - \tilde{m}(t) \leq y\right) = \mathbb{E}e^{-C'Ye^{-\sqrt{2}y}}, \tag{1.8}$$

where $\tilde{m}(t) = \sqrt{2}t - \frac{1}{2\sqrt{2}} \log t$, C' is a constant and Y is a random variable that is the limit of a martingale (but different from Z!).

(ii) The point process

$$\mathcal{E}_t \equiv \sum_{k\leq n(t)} \delta_{x_k(t)-\tilde{m}(t)} \to \mathcal{E}, \tag{1.9}$$

as $t \uparrow \infty$, in law, where

$$\mathcal{E} = \sum_{k,j} \delta_{\eta_k+\sigma_2\Lambda_j^{(k)}}, \tag{1.10}$$

where η_k is the k-th atom of a mixture of Poisson point process with intensity measure $C'Ye^{-\sqrt{2}y}dy$, with C' and Y as in (i), and $\Lambda_i^{(k)}$ are the atoms of independent and identically distributed point processes $\Lambda^{(k)}$, which are the limits in law of

$$\sum_{j\leq n(t)} \delta_{\tilde{x}_i(t)-\max_{j\leq n(t)} \tilde{x}_j(t)}, \tag{1.11}$$

where $\tilde{x}(t)$ is BBM of speed 1 conditioned on $\max_{j\leq n(t)} \tilde{x}_j(t) \geq \sqrt{2}\sigma_2 t$.

To complete the picture, we give the result for the limiting extremal process in the case $\sigma_1 > \sigma_2$. This result is much simpler and totally unsurprising.

Theorem 1.3. *Let $x_k(t)$ be as in Theorem 1.1, but $\sigma_2 < \sigma_1$. Let $\widetilde{\mathcal{E}} \equiv \widetilde{\mathcal{E}}^0$ and $\widetilde{\mathcal{E}}^{(i)}, i \in \mathbb{N}$ be independent copies of the extremal process (1.6) of standard branching Brownian motion. Let*

$$m(t) \equiv \sqrt{2}t(b\sigma_1 + (1-b)\sigma_2) - \frac{3}{2\sqrt{2}}(\sigma_1 + \sigma_2)\log t - \frac{3}{2\sqrt{2}}(\sigma_1 \log b + \sigma_2 \log(1-b)), \quad (1.12)$$

and set

$$\mathcal{E}_t \equiv \sum_{k \le n(t)} \delta_{x_k(t) - m(t)}. \quad (1.13)$$

Then

$$\lim_{t \uparrow \infty} \mathcal{E}_t = \mathcal{E}, \quad (1.14)$$

exists in law, and

$$\mathcal{E} = \sum_{i,j} \delta_{\sigma_1 e_i + \sigma_2 e_j^{(i)}}, \quad (1.15)$$

where $e_i, e_j^{(i)}$ are the atoms of the point processes $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{E}}^{(i)}$, respectively.

Remark 1.4. *In the case $\sigma_1 < 1$, we see that the limiting process depends only on the values of σ_1 (through the martingale Y) and on σ_2 (through the processes of clusters $\sigma_2 \Lambda^{(k)}$). As σ_2 grows, the clusters become spread out, and in the limit $\sigma_2 = \infty$, the cluster processes degenerate to the Dirac mass at zero. Hence, in that case the extremal process is just a mixture of Poisson point processes. When $\sigma_1 = 0$, and $b > 0$, the martingale limit is just an exponential random variable, the limit of the martingale $n(t)e^{-t}$. The case $b = 1$, $\sigma_1 = 0$ corresponds to the random REM, where there is just a random number of iid random variables of variance one present.*

Remark 1.5. *We have decided to write this paper only for the case of two speeds. It is fairly straightforward to extend our results to the general case of piecewise constant speed with a fixed number of change points. The details will be presented in a separate paper [14]. The general case of variable speed still offers more challenges, in spite of recent progress [16, 17].*

1.2 Outline of the proof

The proof of our result follows the strategy used in [3]. The main difference is that we show that particles that will reach the level of the extremes at time t must, at the time of the speed change, tb, lie in a $\sqrt{t}$-neighbourhood of a value $\sqrt{2}(\sigma_2 - 1)bt$ below the straight line of slope $\sqrt{2}$. This is the done in Section 2. Then two pieces of information are needed: in Section 3 we get precise bounds on the probabilities of BBM to reach values at excessively large heights, and more generally we control the behaviour of solutions of the F-KPP equations very much ahead of the travelling wave front. The final results comes from combining this information with the precise distribution of particles at the time of the speed change. This is done in Section 4 by proving the convergence of a certain martingale, analogous, but distinct from the derivative martingale that appears in normal BBM. The identification and the proof of L^1 convergence of this martingale is the key idea. Using this information in Sections 5 and 6, the convergence of the maximums, respectively the Laplace functional of the extremal process are proven, much along the lines on [3]. Section 7 provides various characterisations of the limiting process, as in [3]. In particular, we describe the extremal process in terms of an auxiliary process, constructed from a Poisson point process with a strange intensity to those atoms we add BBM's with negative drift. Interestingly, the process of the cluster extremes of this auxiliary process is again Poisson with random intensity driven by the new martingale. The results stated above follow then from looking at the clusters from their maximal points. In the final Section 8, we give the simple proof of Theorem 1.3

2 Localisation of paths

The key to understanding the behaviour of the two speed BBM is to control the positions of particle at time bt which are in the top at time t. This is done using Gaussian estimates.

Proposition 2.1. *Let $\sigma_1 < \sigma_2$. For any $d \in \mathbb{R}$ and any $\epsilon > 0$, there exists a constant $A > 0$ such that for all t large enough*

$$\mathbb{P}\left[\exists_{j\leq n(t)} \text{ s.t. } x_j(t) > \tilde{m}(t) - d \text{ and } x_j(bt) - \sqrt{2}\sigma_1^2 bt \notin [-A\sqrt{t}, A\sqrt{t}]\right] \leq \epsilon. \tag{2.1}$$

Proof. Using a first order Chebyshev inequality we bound (2.1) by

$$\begin{aligned} & e^t\mathbb{E}\left[\mathbb{1}_{\left\{\sigma_1\sqrt{bt}w_1 - \sqrt{2}\sigma_1^2 bt\notin[-A\sqrt{t},A\sqrt{t}]\right\}}\mathbb{P}_{w_2}\left(\sigma_2\sqrt{(1-b)t}w_2 > \tilde{m}(t) - d - \sigma_1\sqrt{bt}w_1\right)\right] \\ = \quad & e^t\mathbb{E}\Big[\mathbb{1}_{\{w_1-\sqrt{2}\sigma_1\sqrt{bt}\notin[-A',A']\}}\mathbb{P}_{w_2}\Big(w_2 > \tfrac{\sqrt{2t}-\sigma_1\sqrt{b}w_1}{\sigma_2\sqrt{1-b}} - \tfrac{\log t}{2\sqrt{2}\sigma_2\sqrt{(1-b)t}} - \tfrac{d}{\sigma_2\sqrt{(1-b)t}}\Big)\Big] \\ \equiv \quad & (R1) + (R2), \end{aligned} \tag{2.2}$$

where w_1, w_2 are independent $\mathcal{N}(0,1)$-distributed, $A' = \frac{1}{\sqrt{b}\sigma_1}A$, $\mathbb{P}_{w_2}$ denotes the law of the variable w_2. Introducing into the last line the identity in the form

$$1 = \mathbb{1}_{\{\sqrt{2t}-\sigma_1\sqrt{b}w_1<\log t\}} + \mathbb{1}_{\{\sqrt{2t}-\sigma_1\sqrt{b}w_1\geq\log t\}} \tag{2.3}$$

we can write it as $(R1) + (R2)$.

We first show $\lim_{t\to\infty}(R1) = 0$. Using the standard Gaussian tail estimate

$$\int_u^\infty e^{-x^2/2}dx \leq u^{-1}e^{-u^2/2}, \tag{2.4}$$

$(R1)$ is bounded from above by

$$e^t\mathbb{P}\left[\sqrt{2t} - \sigma_1\sqrt{b}w_1 < \log t\right] \leq e^{t(1-1/b\sigma_1^2)+t^{1/2}\log t/b\sigma_1^2} \to 0 \quad \text{as } t\to\infty. \tag{2.5}$$

For (R2) we can use again (2.4) to show that (R2) is smaller than

$$\begin{aligned} & e^t(2\pi)^{-1}\int_{\substack{w-\sqrt{2}\sigma_1\sqrt{bt}\notin[-A',+A'] \\ \sqrt{2t}-\sigma_1\sqrt{b}w_1\geq\log t}} \frac{e^{-w^2/2}}{\frac{\sqrt{2t}}{\sqrt{1-b}\sigma_2} - \frac{\sigma_1\sqrt{b}}{\sigma_2\sqrt{1-b}}w} \\ & \times\exp\left(-\tfrac{1}{2}\left(\tfrac{\sqrt{2t}-\sigma_1\sqrt{b}w-\log t/(2\sqrt{2}\sqrt{t})-d/\sqrt{t}}{\sqrt{1-b}\sigma_2}\right)^2\right)\mathrm{d}w. \end{aligned} \tag{2.6}$$

We change variables $w = \sqrt{2}\sigma_1\sqrt{bt} + z$. Then the integral in (2.6) can be bounded from above by

$$\frac{M}{\sqrt{2\pi\sigma_2^2(1-b)}}\int_{z\notin[-A',A']} e^{-\frac{z^2}{2\sigma_2^2(1-b)}}\mathrm{d}z, \tag{2.7}$$

where M is some positive constant. (2.7) can be made as small as desired by taking A (and thus A') sufficiently large. □

Remark 2.2. *The point here is that since $\sigma_1^2 < \sigma_1$, these particles are way below $\max_{k\leq n(bt)} x_k(bt)$, which is near $\sqrt{2}\sigma_1 bt$. The offspring of these particles that want to be top at time will have to race much faster (at speed $\sqrt{2}\sigma_2^2$, rather than just $\sqrt{2}\sigma_2$) than normal. Fortunately, there are lots of particles to choose from. We will have to control precisely how many.*

We need a slightly finer control on the path of the extremal particle until the time of speed change. To this end we define two sets on the space of paths, $X : \mathbb{R}_+ \to \mathbb{R}$, The first controls that the position of the path is in a certain tube up to time s and the second the position of the particle at time s.

$$\begin{aligned} \mathcal{T}_{s,r} &= \left\{ X \middle| \forall_{0\leq q\leq s} |X(q) - \tfrac{q}{s}X(s)| \leq ((q\wedge(s-q))\vee r)^\gamma \right\} \\ \mathcal{G}_{s,A,\gamma} &= \left\{ X \middle| X(s) - \sqrt{2}\sigma_1^2 s \in [-As^\gamma, +As^\gamma] \right\} \end{aligned} \tag{2.8}$$

Recall [7] that the ancestral path form 0 to $x_k(s)$ can be written as $x_k(q) = \frac{q}{s}x_k(s) + \mathfrak{z}_k(s)$, where $\mathfrak{z}_k$ is a Brownian bridge from 0 to 0 in time s, independent of $x_k(s)$. We need the following simple fact about Brownian bridges.

Lemma 2.3. *Let $\mathfrak{z}(q)$ be a Brownian bridge starting in zero and ending in zero at time s. Then for all $\gamma > 1/2$, the following is true. For all $\epsilon > 0$ there exists r such that*

$$\lim_{s\uparrow\infty} \mathbb{P}\left(|\mathfrak{z}(q)| < ((q\wedge(s-q))\vee r)^\gamma, \forall\, 0\leq q\leq s\right) > 1-\epsilon. \tag{2.9}$$

Proposition 2.4. *Let $\sigma_1 < \sigma_2$. For any $d\in\mathbb{R}, A > 0$, $\gamma > \frac{1}{2}$ and any $\epsilon > 0$, there exists constants $B > 0$ such that, for all t large enough,*

$$\mathbb{P}\left[\exists_{j\leq n(t)} : x_j(t) > \tilde{m}(t) - d \wedge x_j \in \mathcal{G}_{bt,A,\frac{1}{2}} \wedge x_j \notin \mathcal{G}_{b\sqrt{t},B,\gamma}\right] \leq \epsilon. \tag{2.10}$$

Proof. For B and t sufficiently large the probability in (2.10) is bounded from above by

$$\mathbb{P}\left[\exists_{j\leq n(t)} : x_j(t) > \tilde{m}(t) - d \wedge x_j \in \mathcal{G}_{bt,A,\frac{1}{2}} \wedge x_j \notin \mathcal{T}_{bt,r}\right] \tag{2.11}$$

Let w_1 and w_2 be independent $\mathcal{N}(0,1)$-distributed random variables and $\mathfrak{z}$ a Brownian bridge starting in zero and ending in zero at time bt. Using a first moment method as in the proof of Proposition 2.1 together with the independence of the Brownian bridge from its endpoint, one sees that (2.11) is bounded from above by

$$\begin{aligned} & e^t \mathbb{E}\left[\mathbb{1}_{\left\{\sigma_1\sqrt{bt}w_1 - \sqrt{2}\sigma_1^2 bt \in [-A\sqrt{t}, A\sqrt{t}]\right\}} \mathbb{P}_{w_2}\left(\sigma_2\sqrt{(1-b)t}w_2 > \tilde{m}(t) - d - \sigma_1\sqrt{bt}w_1\right)\right] \\ & \times \mathbb{P}\left[\mathfrak{z} \notin \mathcal{T}_{bt,r}\right] < \epsilon, \end{aligned} \tag{2.12}$$

where the last bound follows from Lemma 2.3 (with ϵ replaced by ϵ/M) and the bound (2.7) obtained in the proof of Proposition 2.1. □

Proposition 2.5. *Let $\sigma_1 < \sigma_2$. For any $d\in\mathbb{R}, A, B > 0$, $\gamma > \frac{1}{2}$ and any $\epsilon > 0$, there exists a constant $r > 0$ such that for all t large enough*

$$\begin{aligned} \mathbb{P}\Big[& \exists_{j\leq n(t)} : x_j(t) > \tilde{m}(t) - d \wedge x_j \in \mathcal{G}_{bt,A,\frac{1}{2}} \cap \mathcal{G}_{b\sqrt{t},B,\gamma} \\ & \wedge x_j(b\sqrt{t} + \cdot) - x_j(b\sqrt{t}) \notin \mathcal{T}_{b(t-\sqrt{t}),r}\Big] \leq \epsilon. \end{aligned} \tag{2.13}$$

Proof. The proof of this proposition is essentially identical to the proof of Proposition 2.4. □

3 Asymptotic behaviour of BBM

Let $\tilde{x}(t)$ denote a standard BBM. We are interested in the asymptotic behavior of

$$\mathbb{P}\left[\max_{1\leq i\leq n(t)} \bar{x}_i(t) > x + \sqrt{2}t\right] \tag{3.1}$$

for $x = at + b\sqrt{t}$, $a \in \mathbb{R}_+, b \in \mathbb{R}$. Recall that $\mathbb{P}\left(\max_{k\leq n(t)} \bar{x}_k(t) > x\right)$ is the solution of the F-KPP equation

$$\partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) + (1-u(t,x)) - \sum_{k=1}^{\infty} p_k(1-u(t,x))^k. \tag{3.2}$$

with initial condition $u(0,x) = \mathbb{1}_{x<0}$. We are more generally interested in the behaviour of solutions for such large values of x. The following proposition is an extension of Lemma 4.5 in [3] for these values of x.

Proposition 3.1. *Let u be a solution to the F-KPP equation with initial data satisfying*
(i) $0 \leq u(0,x) \leq 1$;
(ii) for some $h > 0$, $\limsup_{t\to\infty} \frac{1}{t}\log\int_t^{t(1+h)} u(0,y)dy \leq -\sqrt{2}$;
(iii) for some $v > 0$, $M > 0$, $N > 0$, it holds that $\int_x^{x+N} u(0,y)dy > v$ for all $x \leq -M$;
(iv) moreover, $\int_0^\infty u(0,y)ye^{2y}dy < \infty$.
Then we have for $x = at + o(t)$

$$\lim_{t\to\infty} e^{\sqrt{2}x} e^{x^2/2t} t^{1/2} u(t, x+\sqrt{2}t) = C(a), \tag{3.3}$$

where $C(a)$ is a strictly positive constant. The convergence is uniform for a in compact intervals.

Define for $r > 0$ the function $\Psi(r,t,x+\sqrt{2}t)$ by

$$\Psi(r,t,x+\sqrt{2}t) = \frac{e^{-\sqrt{2}x}}{\sqrt{2\pi(t-r)}}\int_0^\infty u(r,y+\sqrt{2}r)e^{\sqrt{2}y}e^{-\frac{(y-x)^2}{2(t-r)}}\left[1-e^{-2y\left(\frac{x+\frac{3}{2\sqrt{2}}\log t}{t-r}\right)}\right]\mathrm{d}y. \tag{3.4}$$

Lemma 3.2. *For $x = at + o(t)$ we have, under the assumptions of Proposition 3.1,*

$$\begin{aligned}&\lim_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}\Psi(r,t,x+\sqrt{2}t)\\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-a^2r/2}u(r,y+\sqrt{2}r)e^{(\sqrt{2}+a)y}\left(1-e^{-2ay}\right)dy \equiv C(r,a).\end{aligned} \tag{3.5}$$

The convergence is uniform for a in a compact set.

Proof. Using (3.4) we have

$$\begin{aligned}&\lim_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}\Psi(r,t,x+\sqrt{2}t)\\ =\ &\lim_{t\to\infty}\frac{\sqrt{t}}{\sqrt{2\pi(t-r)}}e^{x^2/2t}\int_0^\infty u(r,y+\sqrt{2}r)e^{\sqrt{2}y}e^{-\frac{(y-x)^2}{2(t-r)}}\\ &\times\left[1-\exp\left(-2y\left(\frac{x+\frac{3}{2\sqrt{2}}\log t}{t-r}\right)\right)\right]\mathrm{d}y.\end{aligned} \tag{3.6}$$

Next we show that we can use dominated convergence to take the limit $t \to \infty$ into the integral. First, the integrand is bounded by

$$Be^{-a^2r/2}u(r, y + \sqrt{2}r)e^{(\sqrt{2}+a+1)y}, \tag{3.7}$$

where $B > 0$. As was shown by Bramson [6] (and used in [3]), the solution of the F-KPP equation can be bounded by the solution $u^{(2)}(t,x)$ of the linearised F-KPP equation

$$\partial_t u^{(2)} = \frac{1}{2}u^{(2)}_{xx} - u^{(2)} \tag{3.8}$$

with the same initial condition $u^{(2)}(0,x) = u(0,x)$. Moreover there exists y_0 such that for any $x > 0$

$$u^{(2)}(t,x) \le e^t e^{-x^2/2t} e^{y_0 x/t} \tag{3.9}$$

Thus we get that

$$\begin{aligned} & \int_0^\infty Be^{-a^2r/2}u(r, y + \sqrt{2}r)e^{(\sqrt{2}+a+1)y}\mathrm{d}y \\ \le & \int_0^\infty B(r)e^{-a^2r/2}e^{-y^2/2r}e^{(a+1)y}\mathrm{d}y < \infty. \end{aligned} \tag{3.10}$$

where $B(r)$ is a constant that only depends on r. Hence we can apply dominated convergence to (3.6) and obtain

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}}\int_0^\infty u(r, y + \sqrt{2}r)e^{\sqrt{2}y} \lim_{t\to\infty}\left[e^{\sqrt{2}y}e^{-\frac{(y-x)^2}{2(t-r)}}\left[1 - e^{-2y\left(\frac{x+\frac{3}{2\sqrt{2}}\log t}{t-r}\right)}\right]\right]\mathrm{d}y \\ = & \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-a^2r/2}u(r, y + \sqrt{2}r)e^{(\sqrt{2}+a)y}\left(1 - e^{-2ay}\right)\mathrm{d}y. \end{aligned} \tag{3.11}$$

This proves the lemma. □

Proof of Proposition 3.1. Due to the assumptions (i),(ii),(iii) and (iv) we can use Proposition 4.3 of [3] for $t > 8r$ and $x > 8r - \frac{3}{2\sqrt{2}}\log t$ and r large enough:

$$\gamma^{-1}(r)\Psi(r,t,x + \sqrt{2}t) \le u(t, x + \sqrt{2}t) \le \gamma(r)\Psi(r,t,x + \sqrt{2}t), \tag{3.12}$$

where $\gamma(r)$ does not depend x and t and $\lim_{r\to\infty}\gamma(r) = 1$. Since $\gamma(r) \to 1$ as $r \to \infty$ this implies

$$\limsup_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}u(t, x + \sqrt{2}t) \le \liminf_{r\to\infty} C(r,a) \tag{3.13}$$

and

$$\liminf_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}u(t, x + \sqrt{2}t) \ge \limsup_{r\to\infty} C(r,a) \tag{3.14}$$

Hence $\lim_{r\to\infty} C(r,a) = C(a)$ exists. Moreover,

$$\lim_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}u(t, x + \sqrt{2}t) \tag{3.15}$$

exists and is equal to $C(a)$. It is left to show that $C(a) \neq 0$ for $a > 0$. If $C(a) = 0$ we would have

$$\lim_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}u(t, x + \sqrt{2}t) = 0, \tag{3.16}$$

but

$$\lim_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}u(t, x + \sqrt{2}t) \ge C(r,a)\gamma(r)^{-1}, \tag{3.17}$$

for r large enough, by (3.12). This contradicts (3.16). The same proposition implies

$$\lim_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}u(t, x + \sqrt{2}t) \le C(r,a)\gamma(r). \tag{3.18}$$

Hence $C(a) \neq \infty$. Proposition 3.1 is proven. □

4 The McKean martingale

In this section we pick up the idea of [15] and consider a suitable convergent martingale for the time inhomogeneous BBM with $\sigma_1 < \sigma_2$. Let $x_i(s)$, $1 \leq i \leq n(s)$ be the particles of a BBM where the Brownian motions have variance σ_1^2 with $\sigma_1^2 < 1$. Define

$$Y_s = \sum_{i=1}^{n(s)} e^{-s(1+\sigma_1^2)+\sqrt{2}x_i(s)}. \tag{4.1}$$

This turns out to be a uniformly integrable martingale that converges almost surely to a positive limit Y.

Remark 4.1. *Note that in terms of statistical mechanics, Y_s can be thought of as a normalised partition function at inverse temperature $\sigma_1\sqrt{2}$ (for ordinary BBM). Here the critical temperature is $\sqrt{2}$, so that we are in the high-temperature case. In the case of the GREM, where the underlying tree is deterministic, this quantity is known to even converge to a constant [4].*

Theorem 4.2. *The limit $\lim_{s\to\infty} Y_s$ exists almost surely and in L^1, is finite and strictly positive.*

The assertion of this theorem follows immediately from the following proposition.

Proposition 4.3. *If $\sigma_1 < 1$, Y_s is a uniformly integrable martingale with $\mathbb{E}[Y_s] = 1$*

Remark 4.4. *We would like to call this martingale* McKean martingale*, since McKean [18] had originally conjectured that this martingale (with $\sigma_1 = 1$) was the martingale in the representation of the extremal distribution of BBM, which, as Lalley and Sellke showed is wrong as it is actually the derivative martingale that appears there. We find it nice to see that in the time-inhomogeneous case with $\sigma_1 < 1$, KcKean turns out to be right! We will see in the proof that the uniform integrability of this martingale breaks down at $\sigma_1 = 1$.*

Remark 4.5. *Note further that if $\sigma_1 = 0$, then $Y_t = e^{-t}n(t)$ which is well known to converge to an exponential random variable.*

Proof. Clearly,

$$\mathbb{E}[Y_s] = e^s \mathbb{E}\left[e^{-(1+\sigma_1^2)s+\sqrt{2}x_1(s)}\right] = 1. \tag{4.2}$$

Next we show that Y_s is a martingale. Let $0 < r < s$. Then

$$\mathbb{E}[Y_s|\mathcal{F}_r] = \sum_{i=1}^{n(r)} \mathbb{E}\left[\sum_{j=1}^{n_j(s-r)} e^{-s(1+\sigma_1^2)+\sqrt{2}(x_j^i(s-r)+x_i(r))} \mid \mathcal{F}_r\right], \tag{4.3}$$

where for $1 \leq i \leq r$: $\left\{x_j^i(s-r), 1 \leq j \leq n_i(s-r)\right\}$ are the particles of independent BBM's with variance σ_1^2 at time $s-r$. (4.3) is equal to

$$\sum_{i=1}^{n(r)} e^{-r(1+\sigma_1^2)+\sqrt{2}x_i(r)} = Y_r, \tag{4.4}$$

as desired.

It remains to show that Y_s is uniformly integrable. We will write abusively $x_k(r)$ for the ancestor of $x_k(s)$ at time $r \leq s$ and write x_k for the entire ancestral path of $x_k(s)$. Define the truncated variable

$$Y_s^A = \sum_{i=1}^{n(s)} e^{-(1+\sigma_1^2)s+\sqrt{2}x_i(s)} \mathbb{1}_{\{x_i \in \mathcal{G}_{s,A,1/2}, x_i \in \mathcal{T}_{s,r}\}}. \tag{4.5}$$

First $Y_s - Y_s^A \geq 0$, and a simple computation using the independence of $x_k(s)$ and $x_k(r) - \frac{r}{s}x_k(s)$ together with Lemma 2.3 shows that

$$\begin{aligned} \mathbb{E}\left[Y_s - Y_s^A\right] &\leq e^s \int_{-\infty}^{\infty} e^{-(1+\sigma_1^2)s+\sqrt{2s}\sigma_1 x} \mathbb{1}_{\{x-\sqrt{2s}\sigma_1 \notin [-A,A]\}} e^{-x^2/2} \tfrac{dx}{\sqrt{2\pi}} + \epsilon \\ &= \int_{|z|>A} e^{-z^2/2} \tfrac{dz}{\sqrt{2\pi}} + \epsilon, \end{aligned} \tag{4.6}$$

which can be made as small as desired by taking A and r to infinity. The key point is that the the second moment of Y_s^A is uniformly bounded in s.

$$\mathbb{E}\left[(Y_s^A)^2\right] = \mathbb{E}\left[\left(\sum_{i=1}^{n(s)} e^{-(1+\sigma_1^2)s+\sqrt{2}x_i(s)} \mathbb{1}_{\{x_i \in \mathcal{G}_{s,A,1/2} \cap \mathcal{T}_{s,r}\}}\right)^2\right] \equiv (T1) + (T2), \tag{4.7}$$

where

$$\begin{aligned} (T1) &= \mathbb{E}\left[\sum_{i=1}^{n(s)} e^{-2((1+\sigma_1^2)s-\sqrt{2}x_i(s))} \mathbb{1}_{\{x_i \in \mathcal{G}_{s,A,1/2} \cap \mathcal{T}_{s,r}\}}\right] \\ (T2) &= \mathbb{E}\left[\sum_{\substack{i,j=1 \\ i \neq j}} e^{-2(1+\sigma_1^2)s+\sqrt{2}(x_i(s)+x_j(s))} \mathbb{1}_{\{x_i, x_j \in \mathcal{G}_{s,A,1/2} \cap \mathcal{T}_{s,r}\}}\right] \end{aligned} \tag{4.8}$$

We start by controlling $(T1)$.

$$\begin{aligned} (T1) &\leq \frac{e^{(s-2s(1+\sigma_1^2)}}{\sqrt{2\pi}} \int_{\sqrt{2s}\sigma_1 - A/\sigma_1}^{\sqrt{2s}\sigma_1 + A/\sigma_1} e^{2\sqrt{2s}\sigma_1 x} e^{-x^2/2} dx \\ &= \frac{e^{-(1-\sigma_1^2)s}}{\sqrt{2\pi}} \int_{-A/\sigma_1}^{A/\sigma_1} e^{-x^2/2} dx \leq e^{-(1-\sigma_1^2)s} \to 0 \quad \text{as } s \to \infty. \end{aligned} \tag{4.9}$$

Now we control $(T2)$. By the sometimes so-called "many-to-two lemma" (see e.g.[6], Lemma 10), and dropping the useless parts of the conditions on the Brownian bridges

$$\begin{aligned} (T2) \leq \quad & K e^s \int_0^s e^{s-q} \int_{\sqrt{2}\sigma_1^2 q - I_1(q,s)}^{\sqrt{2}\sigma_1^2 q + I_1(q,s)} \left(\int_{\sqrt{2}\sigma_1^2 s - A\sqrt{s} - x}^{\sqrt{2}\sigma_1^2 s + A\sqrt{s} - x} e^{-s(1+\sigma_1^2)+\sqrt{2}(x+y)} \right. \\ & \left. \times e^{-\frac{y^2}{2\sigma_1^2(s-q)}} \frac{dy}{\sigma_1\sqrt{2\pi(s-q)}} \right)^2 e^{-\frac{x^2}{2q\sigma_1^2}} \frac{dx dq}{\sqrt{2\pi\sigma_1^2 q}}, \end{aligned} \tag{4.10}$$

where $K = \sum_{i=1}^{\infty} p_k k(k-1)$ and $I_1(q,s) = Aq/\sqrt{s} + ((q \wedge (s-q)) \vee r)^\gamma$. Moreover We change variables $x = z + \sqrt{2}\sigma_1^2 q$ and obtain

$$\begin{aligned} & K e^s \int_0^s e^{s-q} \int_{-I_1(q,s)}^{+I_1(q,s)} \left(\int_{\sqrt{2}\sigma_1^2(s-q) - A\sqrt{s} - z}^{\sqrt{2}\sigma_1^2(s-q) + A\sqrt{s} - z} e^{-s(1+\sigma_1^2)+\sqrt{2}(z+\sqrt{2}\sigma_1^2 q + y)} \right. \\ & \left. \times e^{-\frac{y^2}{2\sigma_1^2(s-q)}} \frac{dy}{\sigma_1\sqrt{2\pi(s-q)}} \right)^2 e^{-\frac{(z+\sqrt{2}\sigma_1^2 q)^2}{2\sigma_1^2 q}} \frac{dz dq}{\sqrt{2\pi\sigma_1^2 q}}, \end{aligned} \tag{4.11}$$

Now we change variables $w = \frac{y}{\sigma_1\sqrt{s-q}} - \sqrt{2}\sigma_1\sqrt{s-q}$. (4.11) is equal to

$$K \int_0^s e^{-q(1-2\sigma_1^2)} \int_{-I_1(q,s)}^{+I_1(q,s)} e^{+2\sqrt{2}z} \left(\int_{\frac{-A\sqrt{s}-z}{\sigma_1\sqrt{s-q}}}^{\frac{+A\sqrt{s}-z}{\sigma_1\sqrt{s-q}}} e^{-w^2/2} \frac{dw}{\sqrt{2\pi}} \right)^2 e^{-\frac{(z+\sqrt{2}\sigma_1^2 q)^2}{2\sigma_1^2 q}} \frac{dz dq}{\sqrt{2\pi\sigma_1^2 q}}. \tag{4.12}$$

Now the integral with respect to w is bounded by 1. Hence (4.12) is bounded from above by

$$K\int_0^s e^{-q(1-2\sigma_1^2)}\int_{-I_1(q,s)/\sigma_1\sqrt{q}}^{+I_1(q,s)/\sigma_1\sqrt{q}} e^{-\frac{(z-\sqrt{2}\sigma_1\sqrt{q})^2}{2}}\frac{dzdq}{\sqrt{2\pi}}. \tag{4.13}$$

We split the integral over q into the three parts R_1, R_2, and R_3 according to the integration from 0 to r, r to $s-r$, and $s-r$ to r, respectively. Then

$$R_2 \leq K\int_r^{s-r} e^{-q(1-2\sigma_1^2)}\frac{e^{-\frac{1}{2}\left(I_1(q,s)/\sigma_1\sqrt{q}-\sqrt{2}\sigma_1\sqrt{q}\right)^2}}{\sqrt{2\pi}\left(\sqrt{2}\sigma_1\sqrt{q}-I_1(q,s)/\sigma_1\sqrt{q}\right)}dq \tag{4.14}$$

This is bounded by

$$K\int_r^{s-r} e^{-(1-\sigma_1^2)q+O(q^\gamma)}dq \leq \frac{C}{1-\sigma_1^2}e^{-c(1-\sigma_1^2)r}. \tag{4.15}$$

For R_1 the integral over z can only be bounded by one. This gives

$$R_1 \leq K\int_0^r e^{(2\sigma_1^2-1)q}dq \equiv D_1(r), \tag{4.16}$$

R_3 can be treated the same way as R_2 and we get

$$R_3 \leq K\int_{s-r}^{s} e^{-(1-\sigma_1^2)q+O(r^\gamma)}dq \leq \frac{K}{1-\sigma_1^2}e^{-(1-\sigma_1^2)(s-r)+O(r^\gamma)} \to 0 \quad \text{as } s\to\infty. \tag{4.17}$$

Putting all three estimates together, we see that $\sup_s \mathbb{E}\left[\left(Y_s^A\right)^2\right] \leq D_2(r)$. From this it follows that Y_s is uniformly integrable. Namely,

$$\begin{aligned}\mathbb{E}[Y_s\mathbb{1}_{Y_s>z}] &= \mathbb{E}[Y_s^A\mathbb{1}_{Y_s>z}] + \mathbb{E}[(Y_s-Y_s^A)\mathbb{1}_{Y_s>z}] \\ &= \mathbb{E}\left[Y_s^A\mathbb{1}_{Y_s^A>z/2}\right] + \mathbb{E}\left[Y_s^A\left(\mathbb{1}_{Y_s>z}-\mathbb{1}_{Y_s^A>z/2}\right)\right] + \mathbb{E}[(Y_s-Y_s^A)\mathbb{1}_{Y_s>z}].\end{aligned} \tag{4.18}$$

For the first term we have

$$\mathbb{E}\left[Y_s^A\mathbb{1}_{Y_s^A>z/2}\right] \leq \frac{2}{z}\mathbb{E}\left[\left(Y_s^A\right)^2\right] \leq \frac{2}{z}D_2(r). \tag{4.19}$$

For the second, we have

$$\begin{aligned}\mathbb{E}\left[Y_s^A\left(\mathbb{1}_{Y_s>z}-\mathbb{1}_{Y_s^A>z/2}\right)\right] &\leq \mathbb{E}\left[Y_s^A\mathbb{1}_{Y_s-Y_s^A\geq z/2}\mathbb{1}_{Y_s^A\leq z/2}\right] \\ &\leq \frac{z}{2}\mathbb{P}\left[(Y_s-Y_s^A)>z/2\right] \leq \mathbb{E}\left[Y_s-Y_s^A\right].\end{aligned} \tag{4.20}$$

The last term in (4.18) is also bounded by $\mathbb{E}\left[Y_s-Y_s^A\right]$. Choosing now A and r such that $\mathbb{E}\left[Y_s-Y_s^A\right] \leq \epsilon/3$, and then z so large that $\frac{2}{z}D_2(r) \leq \epsilon/3$, we obtain that $\mathbb{E}[Y_s\mathbb{1}_{Y_s>z}] \leq \epsilon$, for large enough z, uniformly in s. Thus Y_s is uniformly integrable, which we wanted to show. □

Proof of Theorem 4.2. By Proposition 4.3 Y_s is a positive uniformly integrable martingale. By Doob's martingale convergence theorem we have that $\lim Y_s = Y$ exists almost surely and is finite. Moreover Y is positive and $Y_s \xrightarrow{L^1} Y$. In particular, this implies $Y \not\equiv 0$. □

We will also need to control the processes $\tilde{Y}_{s,\gamma}^A = \sum_{i=1}^{n(s)} e^{-(1+\sigma_1^2)s+\sqrt{2}x_i(s)}\mathbb{1}_{x_i\in\mathcal{G}_{s,A,\gamma}}$.

Lemma 4.6. *The family of random variables $\tilde{Y}_{s,\gamma}^A$, $s, A \in \mathbb{R}_+, 1>\gamma>1/2$ is uniformly integrable and converges, as $s\uparrow\infty$ and $A\uparrow\infty$, to Y, both in probability and in L^1.*

Proof. The proof of uniform integrability is a rerun of the proof of Proposition 4.3, noting that the bounds on the truncated second moments are uniform in A. Moreover, the same computation as in Eq. (4.6) shows that $\mathbb{E}|Y_s - \tilde{Y}^A_{s,\gamma}| \leq \epsilon$, uniformly in s, for A large enough. Therefore,

$$\lim_{A\uparrow\infty} \limsup_{s\uparrow\infty} \mathbb{E}|Y_s - \tilde{Y}^A_{s,\gamma}| = 0, \tag{4.21}$$

which implies that $Y_s - \tilde{Y}^A_{s,\gamma}$ converges to zero in probability. Since Y_s converges to Y almost surely, we arrive at the second assertion of the lemma. □

5 Convergence of the maximum of two-speed BBM

Using the results established in the last three sections, we show now the convergence of the law the of the maximum of two-speed BBM in the case $\sigma_1 < \sigma_2$.

Theorem 5.1. *Let $\{x_k(t), 1 \leq k \leq n(t)\}$ be the particles of a time inhomogeneous BBM with $\sigma_1 < \sigma_2$ and the normalising assumption $\sigma_1^2 b + \sigma_2^2(1-b) = 1$. Then, with $\tilde{m}(t)$ as in Theorem 1.1,*

$$\lim_{t\to\infty} \mathbb{P}\left[\max_{1\leq k\leq n(t)} x_k(t) - \tilde{m}(t) \leq y\right] = \mathbb{E}\left[\exp\left(-\sigma_2 C(a) Y e^{-\sqrt{2}y}\right)\right]. \tag{5.1}$$

Y is the limit of the McKean martingale from the last section, and $C(a)$ is the positive constant given by

$$C(a) = \lim_{r\to\infty} \int_0^\infty e^{-a^2 r/2} \mathbb{P}\left[\max_{k\leq n(t)} \bar{x}_k(r) > z + \sqrt{2}r\right] e^{(\sqrt{2}+a)z}\left(1 - e^{-2az}\right) dz, \tag{5.2}$$

where $\{\bar{x}_k(t), k \leq n(t)\}$ are the particles of a standard BBM and $a = \sqrt{2}(\sigma_2 - 1)$.

Proof. Denote by $\{x_i(bt), 1 \leq i \leq n(bt)\}$ the particles of a BBM with variance σ_1 at time bt and by $\mathcal{F}_{bt}$ the σ-algebra generated this BBM. Moreover, for $1 \leq i \leq n(bt)$, let $\{x^i_j((1-b)t), 1 \leq j \leq n_i((1-b)t)\}$ denote the particles of independent BBM with variance σ_2 at time $(1-b)t$.

Let us first observe that by the analog of Theorem 1.1. of [10] for two-speed BBM[1] we know that the maximum of our process is not too small, namely that for any $\epsilon > 0$, there exists $d < \infty$, such that

$$\mathbb{P}\left[\max_{1\leq k\leq n(t)} x_k(t) - \tilde{m}(t) \leq -d\right] \leq \epsilon/2. \tag{5.3}$$

Therefore,

$$\begin{aligned} \mathbb{P}\left[-d \leq \max_{1\leq k\leq n(t)} x_k(t) - \tilde{m}(t) \leq y\right] &\leq \mathbb{P}\left[\max_{1\leq k\leq n(t)} x_k(t) - \tilde{m}(t) \leq y\right] \\ &\leq \mathbb{P}\left[-d \leq \max_{1\leq k\leq n(t)} x_k(t) - \tilde{m}(t) \leq y\right] \\ &+ +\epsilon/2 \end{aligned} \tag{5.4}$$

[1] As pointed out in [11], the arguments used for branching random walks carry all over to BBM.

On the other hand, by Proposition 2.1, we have that there exists $A<\infty$, such that

$$\begin{aligned}
&\mathbb{P}\left[\forall_{1\le k\le n(t)}\{-d\le x_k(t)-\tilde m(t)\le y\}\cap\{x_k\in\mathcal{G}_{bt,A,\frac12}\}\right] \qquad (5.5)\\
&\le \mathbb{P}\left[-d\le \max_{1\le k\le n(t)} x_k(t)-\tilde m(t)\le y\right]\\
&= \mathbb{P}\left[\forall_{1\le k\le n(t)}\{-d\le x_k(t)-\tilde m(t)\le y\}\cap\{x_k\in\mathcal{G}_{bt,A,\frac12}\}\ \right]\\
&+\mathbb{P}\left[\exists_{1\le k\le n(t)}\{-d\le x_k(t)-\tilde m(t)\le y\}\cap\{x_k\notin\mathcal{G}_{bt,A,\frac12}\}\right]\\
&\le \mathbb{P}\left[\forall_{1\le k\le n(t)}\{-d\le x_k(t)-\tilde m(t)\le y\}\cap\{x_k\in\mathcal{G}_{bt,A,\frac12}\}\right]+\epsilon/2
\end{aligned}$$

Combining (5.4) and (5.5), we have that

$$\begin{aligned}
&\mathbb{P}\left[\forall_{1\le k\le n(t)}\{-d\le x_k(t)-\tilde m(t)\le y\}\cap\{x_k\in\mathcal{G}_{bt,A,\frac12}\}\right]\\
&\le \mathbb{P}\left[\forall_{1\le k\le n(t)}\{-d\le x_k(t)-\tilde m(t)\le y\}\right] \qquad (5.6)\\
&\le \mathbb{P}\left[\forall_{1\le k\le n(t)}\{-d\le x_k(t)-\tilde m(t)\le y\}\cap\{x_k\in\mathcal{G}_{bt,A,\frac12}\}\right]+\epsilon
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\mathbb{P}\left[\max_{1\le k\le n(t)} x_k(t)-\tilde m(t)\le y\right]\\
&=\mathbb{P}\left[\max_{1\le i\le n(bt)}\max_{1\le j\le n_i((1-b)t)} x_i(bt)+x^i_j((1-b)t)-\tilde m(t)\le y\right]\\
&=\mathbb{E}\left[\prod_{1\le i\le n_i(bt)}\mathbb{P}\left[\max_{1\le j\le n_i((1-b)t)} x^i_j((1-b)t)\le \tilde m(t)-x_i(bt)+y\,|\,\mathcal{F}_{bt}\right]\right]\\
&\le\mathbb{E}\Bigg[\prod_{\substack{1\le i\le n(bt)\\ x_i\in\mathcal{G}_{bt,A,\frac12}}}\mathbb{P}\left[\max_{1\le j\le n_i((1-b)t)}\sigma_2^{-1}x^i_j((1-b)t)\le\sigma_2^{-1}\left(\tilde m(t)-x_i(tb)+y\right)\,|\,\mathcal{F}_{tb}\right]\Bigg]\\
&\quad+\epsilon. \qquad (5.7)
\end{aligned}$$

Of course the corresponding lower bound holds without the ϵ.

Observe that the last probability in (5.7) is equal to

$$1-\mathbb{P}\left[\max_{1\le j\le n_i((1-b)t)}\bar x^i_j((1-b)t)>\sigma_2^{-1}\left(\tilde m(t)-x_i(tb)+y\right)\,|\,\mathcal{F}_{tb}\right], \qquad (5.8)$$

where $\bar x^i_j((1-b)t)$ are the particles of a standard BBM. Using Proposition 3.1 for $(1-b)t$ and $u(t,x)=\mathbb{P}\left(\max\bar x^i_j(t)>x\right)$, and setting

$$C_t(x)\equiv e^{\sqrt2 x+x^2/2t}t^{1/2}u(t,x+\sqrt2 t), \qquad (5.9)$$

we can write the probabilities in the last line of (5.8) as

$$\begin{aligned}
&u\left((1-b)t,\sigma_2^{-1}(\tilde m(t)-x_i(bt)+y)\right) \qquad (5.10)\\
&=C_{(1-b)t}\left(\sigma_2^{-1}(\tilde m(t)-x_i(bt)+y)-t\sqrt2(1-b)\right)\\
&\quad\times e^{-\sqrt2\left(\frac{\tilde m(t)-x_i(bt)+y}{\sigma_2}-\sqrt2(1-b)t\right)}e^{-\frac{1}{2(1-b)t}\left(\frac{\tilde m(t)-x_i(bt)+y}{\sigma_2}-\sqrt2(1-b)t\right)^2}((1-b)t)^{-1/2}
\end{aligned}$$

Now all the $x_i(bt)$ that appear are of the form $x_i(bt)=\sqrt2\sigma_1^2bt+O(\sqrt t)$, so that

$$C_{(1-b)t}\left(\sigma_2^{-1}(\tilde m(t)-x_i(bt)+y)-\sqrt2(1-b)t\right)=C_{(1-b)t}(a(1-b)t+O(\sqrt t)), \qquad (5.11)$$

with (using (1.2))

$$a \equiv \frac{1}{1-b}\left(\frac{\sqrt{2}-\sqrt{2}\sigma_1^2 b}{\sigma_2} - \sqrt{2}(1-b)\right) = \sqrt{2}(\sigma_2 - 1), \tag{5.12}$$

But then, by Proposition 3.1,

$$\lim_{t\uparrow\infty} C_{(1-b)t}\left(\sigma_2^{-1}(\tilde{m}(t) - x_i(bt) + y) - \sqrt{2}(1-b)t)\right) = C(a), \tag{5.13}$$

with uniform convergence for all i appearing in (5.7) and $C(a)$ is the constant given by (5.2). Thus we can rewrite the expectation in (5.7) as

$$\begin{aligned}
&\mathbb{E}\Bigg[\prod_{\substack{1\le i\le n(bt)\\ x_i\in\mathcal{G}_{bt,A,1/2}}} \mathbb{P}\left[\max_{1\le j\le n_i((1-b)t)} \sigma_2^{-1}x_j^i((1-b)t) \le \sigma_2^{-1}\left(\tilde{m}(t) - x_i(tb) + y\right) \mid \mathcal{F}_{tb}\right]\Bigg] \\
&= \mathbb{E}\Bigg[\prod_{\substack{1\le i\le n(bt)\\ x_i\in\mathcal{G}_{bt,A,1/2}}} \Big\{1 - C(a)e^{-\sqrt{2}\left(\frac{\tilde{m}(t)-x_i(bt)+y}{\sigma_2} - \sqrt{2}(1-b)t\right)} \\
&\qquad \times e^{-\frac{1}{2(1-b)t}\left(\frac{\tilde{m}(t)-x_i(bt)+y}{\sigma_2}-\sqrt{2}(1-b)t\right)^2}((1-b)t)^{-1/2}(1+o(1))\Big\}\Bigg].
\end{aligned} \tag{5.14}$$

This is equal to

$$\mathbb{E}\Bigg[\prod_{\substack{1\le i\le n(bt)\\ x_i\in\mathcal{G}_{bt,A,1/2}}} \Big\{1 - C(a)((1-b)t)^{-1/2}e^{(1-b)t - \frac{(\tilde{m}(t)+y-x_i(bt))^2}{2(1-b)t\sigma_2^2}}(1+o(1))\Big\}\Bigg]. \tag{5.15}$$

Using that $x_i(bt) - \sqrt{2}\sigma_1^2 tb \in [-A\sqrt{t}, A\sqrt{t}]$ we have the uniform bounds

$$\exp\left((1-b)t - \tfrac{(\tilde{m}(t)+y-x_i(bt))^2}{2(1-b)t\sigma_2^2}\right) \le \exp\left((1-\sigma_2^2)(1-b)t + \log t + A\sqrt{t}\right). \tag{5.16}$$

Observe that the right-hand side of Eq. (5.16)$\to 0$ as $t\uparrow\infty$, since $\sigma_2^2 > 1$. Hence (5.15) is equal to

$$\mathbb{E}\Bigg[\prod_{\substack{1\le i\le n(bt)\\ x_i\in\mathcal{G}_{bt,A,1/2}}} \exp\left(-C(a)((1-b)t)^{-1/2}e^{(1-b)t-\frac{(\tilde{m}(t)+y-x_i(bt))^2}{2(1-b)t\sigma_2^2}}(1+o(1))\right)\Bigg]. \tag{5.17}$$

Expanding the square in the exponent in the last line and keeping only the relevant terms yields

$$\sqrt{2}y + t\sigma_2^2(1-b) + 2\sigma_1^2 bt - \sqrt{2}x_i(bt) + \frac{\left(\sqrt{2}t\sigma_1^2 b - x_i(bt)\right)^2}{2(1-b)\sigma_2^2 t}. \tag{5.18}$$

The terms up to the last one would nicely combine to produce the McKean martingale as coefficient of $C(a)$. However, the last terms are of order one and cannot be neglected. To deal with them, we split the process at time $b\sqrt{t}$. We write somewhat abusively $x_i(bt) = x_i(b\sqrt{t}) + x_l^{(i)}(b(t-\sqrt{t}))$, where we understand that $x_i(b\sqrt{t})$ is the ancestor at time $b\sqrt{t}$ of the particle that at time t is labeled i if we think backwards from time t, while the labels of the particles at time $b\sqrt{t}$ run only over the different ones, i.e. up to $n(b\sqrt{t})$, if we think in the forward direction. No confusion should occur if this is kept in mind.

Using Proposition 2.4 and Proposition 2.5 we can further localise the path of the particle. Recall the definition of $\mathcal{G}_{s,A,\gamma}$ and $\mathcal{T}_{r,s}$, we rewrite (5.17), up to a term of order ϵ, as

$$\mathbb{E}\Bigg[\prod_{\substack{1\le i\le n(b\sqrt{t})\\ x_i\in\mathcal{G}_{b\sqrt{t},B,\gamma}}}\mathbb{E}\Bigg[\prod_{\substack{1\le l\le n_l^{(i)}(b(t-\sqrt{t}))\\ x_i\in\mathcal{G}_{bt,A,\frac{1}{2}};x_l^{(i)}\in\mathcal{T}_{b(t-\sqrt{t}),r}}}\exp\Bigg(-C(a)((1-b)t)^{-1/2} \times\exp\left((1-b)t-\tfrac{(\tilde{m}(t)+y-x_i(b\sqrt{t})-x_l^{(i)}(b(t-\sqrt{t})))^2}{2(1-b)t\sigma_2^2}\right)(1+o(1))\Bigg)\Big|\mathcal{F}_{\sqrt{t}b}\Bigg]\Bigg]. \tag{5.19}$$

Using that $x_i(b\sqrt{t})+x_l^{(i)}(b(t-\sqrt{t}))-\sqrt{2}\sigma_1^2tb\in[-A\sqrt{t},A\sqrt{t}]$ and $\tilde{m}=\sqrt{2}-\frac{1}{2\sqrt{2}}\log t$, we can re-write the terms multiplying $C(a)$ in (5.19) as

$$\begin{aligned}&\exp\Big(-(1+\sigma_1^2)bt+\sqrt{2}(x_i(b\sqrt{t})+x_l^{(i)}(b(t-\sqrt{t})))-\frac{1}{2}\log(1-b)-\sqrt{2}y\\ &-\tfrac{(x_i(b\sqrt{t})+x_l^{(i)}(b(t-\sqrt{t}))-\sqrt{2}\sigma_1^2bt)^2}{2(1-b)\sigma_2^2t}+O(1/\sqrt{t})\Big)\\ &\equiv E(x_i,x_l^{(i)})=E(x_i(b\sqrt{t}),x_l^{(i)}(b(t-\sqrt{t})))=E(x_i(b\sqrt{t}),x_i(bt)-x_i(b\sqrt{t})).\end{aligned} \tag{5.20}$$

Now (5.19) takes the form

$$\mathbb{E}\Bigg[\prod_{\substack{1\le i\le n(b\sqrt{t})\\ x_i\in\mathcal{G}_{b\sqrt{t},B,\gamma}}}\mathbb{E}\Bigg[\exp\Big\{-\sum_{\substack{1\le l\le n_l^{(i)}(b(t-\sqrt{t}))\\ x_i\in\mathcal{G}_{bt,A,\frac{1}{2}};x_l^{(i)}\in\mathcal{T}_{r,b(t-\sqrt{t})}}}C(a)E(x_i,x_l^{(i)})(1+o(1))\Big\}\Big|\mathcal{F}_{b\sqrt{t}}\Bigg]\Bigg]. \tag{5.21}$$

Using the inequalities

$$1-x\le e^{-x}\le 1-x+\frac{1}{2}x^2,\quad x>0, \tag{5.22}$$

for

$$x=\sum_{\substack{1\le l\le n_l^{(i)}(b(t-\sqrt{t}))\\ x_i\in\mathcal{G}_{bt,A,\frac{1}{2}};x_l^{(i)}\in\mathcal{T}_{r,b(t-\sqrt{t})}}}C(a)E(x_i,x_l^{(i)})(1+o(1)) \tag{5.23}$$

we are able to bound (5.21) from below and above. First,

$$\mathbb{E}[x^2|\mathcal{F}_{b\sqrt{t}}]\le e^{-2(1+\sigma_1^2)b\sqrt{t}+2\sqrt{2}x_i(b\sqrt{t})-2\sqrt{2}y}\mathbb{E}\left[\left(Y^A_{b(t-\sqrt{t})}\right)^2\right](1+o(1)), \tag{5.24}$$

where $Y^A_{b(t-\sqrt{t})}$ is the truncated McKean martingale defined in (4.1). Note that its second moment is bounded by $D_2(r)$ (see (4.19)). Second, computing the conditional expectation given $\mathcal{F}_{b\sqrt{t}}$ yields, up to factors $1+o(1)$,

$$\begin{aligned}\mathbb{E}[x|\mathcal{F}_{b\sqrt{t}}] &= \mathbb{E}\Big[\sum_{\substack{1\le l\le n_l^{(i)}(b(t-\sqrt{t}))\\ x_i\in\mathcal{G}_{bt,A,\frac{1}{2}};x_l^{(i)}\in\mathcal{T}_{r,b(t-\sqrt{t})}}}C(a)E(x_i,x_l^{(i)})\Big|\mathcal{F}_{b\sqrt{t}}\Big] \\ &\le e^{b(\sigma_1^2t-\sqrt{t})-\sqrt{2}y}\int_{K_t-A\sqrt{t}}^{K_t+A\sqrt{t}}e^{\sqrt{2}(z+x_i(b\sqrt{t}))-\frac{(z+x_i(b\sqrt{t})-\sqrt{2}\sigma_1^2bt)^2}{2\sigma_2^2(1-b)t}}\frac{e^{-z^2/2\sigma_1^2b(t-\sqrt{t})}dz}{\sqrt{2\pi\sigma_1^2b(t-\sqrt{t})}}\end{aligned} \tag{5.25}$$

where $K_t = \sqrt{2}tb\sigma_1^2 - x_i(b\sqrt{t})$. Performing the change of variables $z = w + K_t$ this is equal to

$$\begin{aligned}
& e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})-\frac{1}{2}\log(1-b)-\sqrt{2}y}\int_{-A\sqrt{t}}^{A\sqrt{t}} e^{-\frac{w^2}{2\sigma_1^2 b(t-\sqrt{t})}-\frac{w^2}{2\sigma_2^2(1-b)t}}\frac{dw}{\sqrt{2\pi\sigma_1^2 b(t-\sqrt{t})}}(1+o(1)) \\
= \; & e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})-\frac{1}{2}\log(1-b)-\sqrt{2}y}\left(\tfrac{\sigma_2^2(1-b)}{1-\sigma_1^2 b/\sqrt{t}}\right)^{1/2}\int_{-A\sqrt{t}}^{A\sqrt{t}} e^{-w^2/2t}\frac{dw}{\sqrt{2\pi t}}(1+o(1)) \\
= \; & e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})-\sqrt{2}y}\left(\tfrac{\sigma_2^2}{1-\sigma_1^2 b/\sqrt{t}}\right)^{1/2}(1-\epsilon)(1+o(1)), \qquad (5.26)
\end{aligned}$$

where $o(1) \leq O(t^{\gamma-1})$. Using Lemma 2.3 together with the independence of the Brownian bridge from its endpoint, we obtain that the right hand side of (5.26) multiplied by an additional factor $(1-\epsilon)$ is also a lower bound. Comparing this to (5.27), one sees that

$$\frac{\mathbb{E}[x^2|\mathcal{F}_{b\sqrt{t}}]}{\mathbb{E}[x|\mathcal{F}_{b\sqrt{t}}]} \leq D_2(r)e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})} \leq Ce^{-(1-\sigma_1^2)b\sqrt{t}+0(t^{\gamma/2})}, \qquad (5.27)$$

which tends to zero uniformly as $t \uparrow \infty$. Thus the second moment term is negligible. Hence we only have to control

$$\begin{aligned}
& \mathbb{E}\left[\prod_{\substack{1\leq i\leq n(b\sqrt{t}) \\ x_i\in\mathcal{G}_{b\sqrt{t},B,\gamma}}}\left(1-C(a)e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})-\sqrt{2}y}\left(\tfrac{\sigma_2^2}{1-\sigma_1^2 b/\sqrt{t}}\right)^{1/2}\right)\right] \\
= \; & \mathbb{E}\left[\exp\left(-\sum_{\substack{1\leq i\leq n(b\sqrt{t}) \\ x_i\in\mathcal{G}_{b\sqrt{t},B,\gamma}}} C(a)e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})-\sqrt{2}y}\left(\tfrac{\sigma_2^2}{1-\sigma_1^2 b/\sqrt{t}}\right)^{1/2}\right)(1+o(1))\right] \\
= \; & \mathbb{E}\left[\exp\left(-C(a)\left(\tfrac{\sigma_2^2}{1-\sigma_1^2 b/\sqrt{t}}\right)^{1/2}e^{-\sqrt{2}y}\tilde{Y}^B_{b\sqrt{t},\gamma}\right)(1+o(1))\right] \qquad (5.28)
\end{aligned}$$

where

$$\tilde{Y}^B_{b\sqrt{t},\gamma} = \sum_{i=1}^{n(b\sqrt{t})} e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})}\mathbb{1}_{x_i(b\sqrt{t})-\sqrt{2}\sigma_1^2 b\sqrt{t}\in[-Bt^{\gamma/2},Bt^{\gamma/2}]}. \qquad (5.29)$$

Now from Lemma 4.6, $\tilde{Y}^B_{b\sqrt{t},\gamma}$ converges in probability and in L^1 to the random variable Y, when we let first t and then B tend to infinity. Since $Y^B_{b\sqrt{t},\gamma} \geq 0$ and $C(a) > 0$, it follows

$$\begin{aligned}
& \lim_{B\uparrow\infty}\liminf_{t\uparrow\infty}\mathbb{E}\left[\exp\left(-C(a)\left(\tfrac{\sigma_2^2}{1-\sigma_1^2 b/\sqrt{t}}\right)^{1/2}\tilde{Y}^B_{b\sqrt{t},\gamma}e^{-\sqrt{2}y}\right)\right] \\
= \; & \lim_{B\uparrow\infty}\limsup_{t\uparrow\infty}\mathbb{E}\left[\exp\left(-\sigma_2 C(a)\tilde{Y}^B_{b\sqrt{t},\gamma}e^{-\sqrt{2}y}\right)\right] \\
= \; & \mathbb{E}\left[\exp\left(-\sigma_2 C(a)Ye^{-\sqrt{2}y}\right)\right]. \qquad (5.30)
\end{aligned}$$

Finally, letting r tend to $+\infty$, all the ϵ-errors (that are still present implicitly, vanish. This concludes the proof of Theorem 5.1. □

6 Existence of the limiting process

The following existence theorem is the basic step in the proof of Theorem 1.1.

Theorem 6.1. *Let $\sigma_1 < \sigma_2$. Then, the point processes $\mathcal{E}_t = \sum_{k\leq n(t)}\delta_{x_k(t)-\tilde{m}(t)}$ converges in law to a non-trivial point process $\mathcal{E}$.*

Proof. It suffices to show that, for $\phi \in \mathcal{C}_c(\mathbb{R})$ positive, the Laplace functional

$$\Psi_t(\phi) = \mathbb{E}\left[\exp\left(-\int \phi(y)\mathcal{E}_t(dy)\right)\right], \tag{6.1}$$

of the processes $\mathcal{E}_t$ converges. First observe that this limit cannot be zero, since the maximum of the time inhomogeneous BBM converges by Theorem 5.1. As for standard BBM (see e.g. [3]), it follows

$$\lim_{N\to\infty}\lim_{t\to\infty}\mathbb{P}\left[\mathcal{E}_t(B) > N\right] = 0, \text{ for any bounded } B \subset \mathbb{R}, \tag{6.2}$$

which implies the locally finiteness of the limiting point process. As in [3] we decompose

$$\Psi_t(\phi) = \Psi_t^{<\delta}(\phi) + \Psi_t^{>\delta}(\phi), \tag{6.3}$$

where

$$\begin{aligned} \Psi_t^{<\delta}(\phi) &= \mathbb{E}\left[\exp\left(-\int \phi(y)\mathcal{E}_t(dy)\right)\mathbb{1}_{\max\mathcal{E}_t\le\delta}\right] \\ \Psi_t^{>\delta}(\phi) &= \mathbb{E}\left[\exp\left(-\int \phi(y)\mathcal{E}_t(dy)\right)\mathbb{1}_{\max\mathcal{E}_t>\delta}\right]. \end{aligned} \tag{6.4}$$

Here we write shorthand $\max\mathcal{E}_t \le \delta$ for $\max_{k\le n(t)}(x_k(t) - m(t) \le \delta$. By Theorem 5.1 we have

$$\limsup_{\delta\to\infty}\limsup_{t\to\infty}\Psi_t^{>\delta}(\phi) \le \limsup_{\delta\to\infty}\limsup_{t\to\infty}\mathbb{P}[\max\mathcal{E}_t > \delta] = 0. \tag{6.5}$$

Hence it remains to analyse the behaviour of $\Psi_t^{<\delta}(\phi)$. We claim that

$$\lim_{\delta\to\infty}\lim_{t\to\infty}\Psi_t^{<\delta}(\phi) = \Psi(\phi) \tag{6.6}$$

exists and is strictly smaller than 1. To see this set

$$\bar\phi(z) = \phi(\sigma_2 z) \tag{6.7}$$

and

$$g_\delta(z) = e^{-\bar\phi(-z)}\mathbb{1}_{\{-z\sigma_2\le\delta\}}. \tag{6.8}$$

Moreover, define

$$u_\delta(t,z) = 1 - \mathbb{E}\left[\prod_{j\le n(t)} g_\delta(z - \bar{x}_j(t))\right]. \tag{6.9}$$

where $\{\bar{x}_j(t), 1 \le j \le n(t)\}$ are the particles of a standard BBM with variance 1. We observe that by [18] $u_\delta(t,x)$ solves the F-KPP equation (3.2) with initial condition $u_\delta(0,x) = 1 - g_\delta(x)$. Next we verify Assumptions (i)-(iv) of Proposition 3.1. (i) is clear. Moreover, $g_\delta(x) = 1$ for x large enough in the positive , and $g_\delta(x) = 0$ for $-x$ large enough, so that Conditions (ii)-(iv) of Proposition 3.1 are satisfied. Now

$$\begin{aligned} \Psi_t^{<\delta}(\phi) &= \mathbb{E}\left[\prod_{i\le n(bt)}\mathbb{E}\left[\prod_{x_j^i\le n_i((1-b)t)} g_\delta((\tilde{m}(t) - x_i(bt) - x_j^i((1-b)t))/\sigma_2)\middle|\mathcal{F}_{bt}\right]\right] \\ &= \mathbb{E}\left[\prod_{i\le n(bt)}\mathbb{E}\left[\prod_{\bar{x}_j^i\le n_i((1-b)t)} g_\delta((\tilde{m}(t) - x_i(bt))/\sigma_2 - \bar{x}_j^i((1-b)t))\middle|\mathcal{F}_{bt}\right]\right], \end{aligned} \tag{6.10}$$

where for each i, $\bar{x}^i_j$ are the particles of iid standard BBMs. By Proposition 3.1 and the same calculations as in the proof Theorem 5.1 we have that this converges, as $t \to \infty$, to

$$\mathbb{E}\left[\exp\left(-\sigma_2 C(a,\phi,\delta)Y\right)\right], \tag{6.11}$$

where $C(a,\phi,\delta)$ is the constant that appears in Lemma 3.2, with initial condition $g_\delta(z)$, i.e.

$$C(a,\phi,\delta) = \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty u_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2}(1-e^{-2za}) dz, \tag{6.12}$$

where $a = \sqrt{2}(\sigma_2 - 1)$ and u_δ is the solution to the F-KPP equation (3.2) with initial condition $u_\delta(0,z) = 1 - e^{-\bar{\phi}(z)} \mathbb{1}_{\{z\sigma_2 \le \delta\}}$. Thus the limit $\lim_{t\to\infty} \Psi_t^{<\delta}(\phi) = \Psi^{<\delta}(\phi)$ exists. The limit $\delta \uparrow \infty$ then exists by the same argument as in the proof of Theorem 3.1 of [3]: the function

$$\delta \to \Psi^{<\delta}(\phi) \tag{6.13}$$

is increasing and bounded, Moreover, the maximum is an atom of $\mathcal{E}_t$ and ϕ is nonnegative, and so

$$\Psi_t^{<\delta}(\phi) \le \mathbb{E}\left[\exp\left(-\phi(\max \mathcal{E}_t)\right) \mathbb{1}_{\{\max \mathcal{E}_t \le \delta\}}\right] \tag{6.14}$$

The limit as $t \to \infty$ and $\delta \to \infty$ of the right hand side of (6.14) exists by Theorem 5.1. Hence

$$\Psi(\phi) = \lim_{\delta\to\infty} \Psi^\delta(\phi) < 1, \tag{6.15}$$

by monotone convergence. This implies the existence of the limiting process. □

Proposition 6.2. *Let $v(t,x)$, $v_\delta(t,x)$ be solutions of the F-KPP equation with initial data $v(0,x) = 1 - e^{-\bar{\phi}(-x)}$ and $v_\delta(0,x) = 1 - e^{-\bar{\phi}(-x)} \mathbb{1}_{\{-x\sigma_2 \le \delta\}}$ respectively. Set*

$$C(a,\phi,\delta) = \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} \left(1 - e^{-2az}\right) dz \tag{6.16}$$

Then $\lim_{\delta\to\infty} C(a,\phi,\delta)$ exists and is given by

$$C(a,\phi) = \lim_{\delta\to\infty} C(a,\phi,\delta) = \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz. \tag{6.17}$$

Moreover,

$$\lim_{t\to\infty} \Psi_t(\phi) = \mathbb{E}\left[\exp\left(-\sigma_2 C(a,\phi)Y\right)\right]. \tag{6.18}$$

Proof. First we note that

$$C(a,\phi.\delta) = \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz. \tag{6.19}$$

To see this, note that for any $K < \infty$,

$$\lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^K v_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz \le \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} K e^{-a^2 t/2} e^{(\sqrt{2}+a)K} = 0. \tag{6.20}$$

Obviously,

$$C(a,\phi,\delta) \le \liminf_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz. \tag{6.21}$$

Due to (6.20), for any $K < \infty$,

$$\begin{aligned} &C(a,\phi,\delta) - \limsup_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz \\ &\ge -e^{-aK} \limsup_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz. \end{aligned} \tag{6.22}$$

Since this holds for all K, and since the finiteness of the limsup in (6.22) follows from the finiteness of $C(a,\phi,\delta)$, we also have that

$$C(a,\phi,\delta) \geq \limsup_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v_\delta(t,z+\sqrt{2}t)e^{(\sqrt{2}+a)z-a^2t/2}dz, \tag{6.23}$$

and Eq. (6.19) follows. It remains to control the limit as $\delta \uparrow \infty$ of the right-hand side of (6.19). But an exact rerun of the proof of Lemma 4.10 in [3] using Lemma 6.4 below instead of Lemma 4.8 of [3] yields that

$$\lim_{\delta\uparrow\infty}\lim_{t\uparrow\infty} \int_0^\infty v_\delta(t,x+\sqrt{2}t)e^{(\sqrt{2}+a)z-a^2t/2}dz \equiv \lim_{\delta\uparrow\infty} F(\delta) \equiv F \tag{6.24}$$

exists. By (6.11) and (6.24) we have

$$\lim_{t\to\infty} \Psi_t^{<\delta}(\phi) = \mathbb{E}\left[\exp\left(-\sigma_2 C(a,\phi,\delta)Y\right)\right] = \mathbb{E}\left[\exp\left(-\frac{\sigma_2}{\sqrt{2\pi}}F(\delta)Y\right)\right]. \tag{6.25}$$

This converges for $\delta \to \infty$ to

$$\mathbb{E}\left[\exp\left(-\frac{\sigma_2}{\sqrt{2\pi}}FY\right)\right]. \tag{6.26}$$

Hence $F = 0$ would imply

$$\lim_{\delta\to\infty}\lim_{t\to\infty} \Psi_t(\phi) = 1, \tag{6.27}$$

which contradicts (6.15) and Theorem 6.1. Hence $F > 0$. Moreover, (6.26) implies (6.18), which concludes the proof of Proposition 6.2. □

We recall the following estimate for the tail probabilities of standard BBM.

Lemma 6.3 ([2], Corollary 10)**.** *There exists $t_0 < \infty$, such that for $z > 1$ and $t \geq t_0$*

$$\mathbb{P}\left[\max_{k\leq n(t)} \bar{x}_k(t) - \sqrt{2}t + \frac{3}{2\sqrt{2}}\log t \geq z\right] \leq \rho z \exp\left(-\sqrt{2}z - \frac{z^2}{2t} + \frac{3z}{2\sqrt{2}}\frac{\log t}{t}\right), \tag{6.28}$$

for some constant $\rho > 0$.

Lemma 6.4. *Let u be a solution of the F-KPP equation with initial data satisfying Assumptions (i)-(iv) of Proposition 3.1. Let*

$$C(a) = \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty u(t,z+\sqrt{2}t)e^{(\sqrt{2}+a)z-a^2t/2}dz. \tag{6.29}$$

Then for any $x \in \mathbb{R}$:

$$\lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty u(t,x+z+\sqrt{2}t)e^{(\sqrt{2}+a)z-a^2t/2}dz = C(a)e^{-(\sqrt{2}+a)x}. \tag{6.30}$$

Moreover, for any bounded continuous function $h(x)$, that is zero for x small enough

$$\begin{aligned} &\lim_{t\to\infty} \int_{-\infty}^0 \mathbb{E}\left[h\left(y+\max \bar{x}_i(t)-\sqrt{2}t\right)\right] \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{2}+a)y-a^2t/2}dy \\ =\ & C(a)\int_{\mathbb{R}} h(z)(\sqrt{2}+a)e^{-(\sqrt{2}+a)z}dz, \end{aligned} \tag{6.31}$$

where $\{\bar{x}_i(t), i \leq n(t)\}$ are the particles of a standard BBM with variance 1. Here $C(a)$ is the constant from (6.29) for u satisfying the initial condition $\mathbb{1}_{\{x\leq 0\}}$.

Proof. We have by a simple change of variables

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}\int_0^\infty u(t,z+\sqrt{2}t)e^{(\sqrt{2}+a)z-a^2t/2}dz \\ =\;&\frac{e^{(\sqrt{2}+a)x}}{\sqrt{2\pi}}\int_{-x}^\infty u(t,x+z+\sqrt{2}t)e^{(\sqrt{2}+a)z-a^2t/2}dz.\end{aligned} \tag{6.32}$$

Moreover, $\lim_{t\to\infty} u(t,x+z+\sqrt{2}t)=0$ implies

$$\lim_{t\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-x}^0 u(t,x+z+\sqrt{2}t)e^{(\sqrt{2}+a)z-a^2t/2}dz=0, \tag{6.33}$$

which proves (6.30). Moreover, (6.30) with initial condition $\mathbb{1}_{\{x\le 0\}}$ implies that (6.31) holds for $h(x)=\mathbb{1}_{[b,\infty)}, b\in\mathbb{R}$. For general h (6.31) follows in the same way as Lemma 4.11 in [3] by linearity and a monotone class argument. □

7 The auxiliary process

We define the following auxiliary process that has the same limiting behaviour as that of the two-speed BBM. We will denote the law of these processes by P and expectations by E. If desired, all ingredients of the auxiliary process can be thought of to be defined on a new probability space. Let $(\eta_i; i\in\mathbb{N})$ be the atoms of a Poisson point process η on $(-\infty,0)$ with intensity measure

$$\tfrac{\sigma_2}{\sqrt{2\pi}}e^{-(\sqrt{2}+a)z}e^{-a^2t/2}dz. \tag{7.1}$$

For each $i\in\mathbb{N}$ consider independent standard BBMs $\bar{x}^i$. The auxiliary point process of interest is the superposition of the i.i.d BBMs with drift shifted by $\eta_i+\frac{1}{\sqrt{2}+a}\log Y$, where a is the constant defined in (5.12):

$$\Pi_t=\sum_{i,k}\delta_{\left(\eta_i+\frac{1}{\sqrt{2}+a}\log Y+\bar{x}^i_k(t)-\sqrt{2}t\right)\sigma_2}. \tag{7.2}$$

Remark 7.1. *The form of the auxiliary process is similar to the case of standard BBM, but with a different intensity of the Poisson process. In particular, the intensity decays exponentially with t. This is a consequence of the fact that particles at the time of the speed change were forced to be $O(t)$ below the line $\sqrt{2}t$, in contrast to the $O(\sqrt{t})$ in the case of ordinary BBM. The reduction of the intensity of the process with t forces the particles to be selected at these locations.*

Theorem 7.2. *Let $\mathcal{E}_t$ be the extremal process of the two-speed BBM. Then*

$$\lim_{t\to\infty}\mathcal{E}_t \overset{law}{=} \lim_{t\to\infty}\Pi_t. \tag{7.3}$$

Proof. Using the notation $\bar\phi(z)=\phi(\sigma_2 z)$ and by the form of the Laplace functional of a Poisson point process we have

$$\begin{aligned}&E\left[\exp\left(-\int\phi(x)\Pi_t(dx)\right)\right]\\ =\;&E\left[\exp\left(-\sigma_2\int_{-\infty}^0\left\{1-E\left[\exp\left(-\sum_{k\le n(t)}\bar\phi\left(z+\bar{x}_k(t)-\sqrt{2}t+\frac{\log Y}{\sqrt{2}+a}\right)\right)\right]\right\}\right.\right.\\ &\left.\left.\times e^{-(\sqrt{2}+a)z}e^{-a^2t/2}dz\right)\right]\\ =\;&E\left[\exp\left(\frac{\sigma_2}{\sqrt{2\pi}}\int_0^\infty u\left(t,z+\sqrt{2}t-\frac{1}{\sqrt{2}+a}\log Y\right)e^{(\sqrt{2}+a)z}e^{-a^2t/2}dz\right)\right],\end{aligned} \tag{7.4}$$

with

$$u(t,x) = 1 - E\left[\exp\left(-\sum_{k\le n(t)} \bar\phi(-x+\bar x_k(t))\right)\right]. \tag{7.5}$$

By Lemma 6.4 we have

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty u\left(t, z+\sqrt{2}t - \frac{1}{\sqrt{2}+a}\log Y\right) e^{(\sqrt{2}+a)z} e^{-a^2t/2} dz \\ = \;& Y \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty u(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z} e^{-a^2t/2} dz, \end{aligned} \tag{7.6}$$

which exists and is strictly positive by Proposition 6.2. This implies that the Laplace functionals of $\lim_{t\to\infty}\Pi_t$ and of the extremal process of the two-speed BBM are equal. □

The following proposition shows that in spite of the different Poisson ingredients, when we look at the process of the extremes of each of the $x^i(t)$, we end up with a Poisson point process just like in the standard BBM case.

Proposition 7.3. *Define the point process*

$$\Pi_t^{ext} \equiv \sum_i \delta_{\left(\eta_i + \frac{1}{\sqrt{2}+a}\log Y + \max_{k\le n_i(t)} \bar x_k^i(t) - \sqrt{2}t\right)\sigma_2}. \tag{7.7}$$

Then

$$\lim_{t\to\infty} \Pi_t^{ext} \stackrel{law}{=} P_Y \equiv \sum_{i\in\mathbb{N}} \delta_{p_i}, \tag{7.8}$$

where P_Y is the Poisson point process on $\mathbb{R}$ with intensity measure $\sigma_2 C(a) Y \sqrt{2} e^{-\sqrt{2}x}dx$.

Proof. We consider the Laplace functional of Π_t^{ext}. Let $M^{(i)}(t) = \max \bar x_k^{(i)}(t)$ and as before $\bar\phi(z) = \phi(\sigma_2 z)$. We want to show

$$\begin{aligned} &\lim_{t\uparrow\infty} E\left[\exp\left(-\sum_i \bar\phi(\eta_i + M^{(i)}(t) - \sqrt{2}t\right)\right] \\ = \;& \exp\left(-\sigma_2 C(a)\int_{-\infty}^{\infty}\left(1-e^{-\phi(x)}\right)\sqrt{2}e^{-\sqrt{2}x}dx\right). \end{aligned} \tag{7.9}$$

Since η_i is a Poisson point process and the $M^{(i)}$ are i.i.d. we have

$$\begin{aligned} &E\left[\exp\left(-\sum_i \bar\phi(\eta_i + M^{(i)}(t) - \sqrt{2}t\right)\right] \\ = \;& \exp\left(-\sigma_2 \int_{-\infty}^0 E\left[1 - e^{-\bar\phi(z+M(t)-\sqrt{2}t)}\right] e^{-(\sqrt{2}+a)z - a^2t/2} \frac{dz}{\sqrt{2\pi}}\right), \end{aligned} \tag{7.10}$$

where $M(t)$ has the same distribution as one the variables $M^{(i)}(t)$. Now we apply Lemma 6.4 with $h(x) = 1 - e^{-\bar\phi(z)}$. Hence the result follows by using that $\bar\phi(z) = \phi(\sigma_2 z)$ and $\sqrt{2}+a = \sqrt{2}\sigma_2$ together with the change of variables $x = \sigma_2 z$. □

The following proposition states that the Poisson points of the auxiliary process contribute to the limiting process come from a neighbourhood of $-at$.

Proposition 7.4. *Let $z \in \mathbb{R}, \epsilon > 0$. Let η_i be the atoms of a Poisson point process with intensity measure $Ce^{-(\sqrt{2}+a)x - a^2t/2}dx$ on $(-\infty, 0]$. Then there exists $B < \infty$ such that*

$$\sup_{t\ge t_0} P\left(\exists i,k : \eta_i + \bar x_k^i(t) - \sqrt{2}t \ge z, \eta_i \notin [-at - B\sqrt{t}, -at + B\sqrt{t}]\right) \le \epsilon. \tag{7.11}$$

Proof. By a first order Chebychev inequality we have

$$\begin{aligned}
& P\left(\exists i,k:\eta_i+\bar{x}_k^{(i)}(t)-\sqrt{2}t\geq z,\eta_i>-at+B\sqrt{t}\right)\\
\leq\ & C\int_{-at+B\sqrt{t}}^{0}P\left(\max\bar{x}_k(t)\geq\sqrt{2}t-x+z\right)e^{-(\sqrt{2}+a)x}e^{-a^2t/2}dx\\
=\ & C\int_{0}^{at-B\sqrt{t}}P\left(\max\bar{x}_k(t)\geq\sqrt{2}t+x+z\right)e^{(\sqrt{2}+a)x}e^{-a^2t/2}dx, \qquad (7.12)
\end{aligned}$$

by the change of variables $x\to -x$. Using the asymptotics of Lemma 6.3 we can bound (7.12) from above by

$$\begin{aligned}
& \rho C\int_0^{at-B\sqrt{t}}t^{-1/2}e^{-\sqrt{2}(x+z)}e^{-(x+z)^2/2t}e^{(\sqrt{2}+a)x}e^{-a^2t/2}dx\\
\leq\ & \rho C\int_{-a\sqrt{t}}^{-B}e^{z^2/2}dz(1+o(1)), \qquad (7.13)
\end{aligned}$$

by changing variables $x\to x/\sqrt{t}-a\sqrt{t}$. This is a Gaussian integral and can be made as small as we want by choosing B large enough. Similarly one bounds

$$P\left(\exists i,k:\eta_i+x_k^i(t)-\sqrt{2}t\geq z,\eta_i<-at-B\sqrt{t}\right)\leq C\rho\int_B^{\infty}e^{z^2/2}dz(1+o(1)). \qquad (7.14)$$

This concludes the proof. □

The next proposition describes the law of the clusters $\bar{x}_k^{(i)}$. This is analogous to Theorem 3.4 in [3].

Proposition 7.5. *Let $x=at+o(t)$ and $\{\tilde{x}_k(t),k\leq n(t)\}$ be a standard BBM under the conditional law $P\left(\cdot\middle|\{\max\tilde{x}_k(t)-\sqrt{2}t-x>0\}\right)$. Then the point process*

$$\sum_{k\leq n(t)}\delta_{\tilde{x}_k(t)-\sqrt{2}t-x} \qquad (7.15)$$

converges in law under $P\left(\cdot\middle|\{\max\tilde{x}_k(t)-\sqrt{2}t-x>0\}\right)$ as $t\to\infty$ to a well defined point process $\bar{\mathcal{E}}$. The limit does not depend on $x-at$ and the maximum of $\bar{\mathcal{E}}$ shifted by x has the law of an exponential random variable with parameter $\sqrt{2}+a$.

Proof. Set $\bar{\mathcal{E}}_t=\sum_k\delta_{\tilde{x}_k(t)-\sqrt{2}t}$ and $\max\bar{\mathcal{E}}_t=\max\tilde{x}_k(t)-\sqrt{2}t$. First we show that for $X>0$

$$\lim_{t\to\infty}P\left(\max\bar{\mathcal{E}}_t>X+x|\max\bar{\mathcal{E}}_t>x\right)=e^{-(\sqrt{2}+a)X}. \qquad (7.16)$$

To see this we rewrite the conditional probability as $\frac{P\left[\max\bar{\mathcal{E}}_t>X+x\right]}{P\left[\max\bar{\mathcal{E}}_t>x\right]}$ and use the uniform bounds of Proposition 4.3 in [3]. Observing that

$$\lim_{t\to\infty}\frac{\Psi(r,t,X+x+\sqrt{2}t)}{\Psi(r,t,x+\sqrt{2}t)}=e^{-(\sqrt{2}+a)X}, \qquad (7.17)$$

where Ψ is defined in Equation (3.4), we get (7.16) by first taking $t\to\infty$ and then $r\to\infty$. The general claim of Proposition 7.5 follows in exactly the same way from (7.16) as Theorem 3.4. in [3]. □

Define the gap process

$$D_t=\sum_k\delta_{\tilde{x}_k(t)-\max_j\tilde{x}_j(t)}. \qquad (7.18)$$

Denote by ξ_i the atoms of the limiting process $\bar{\mathcal{E}}$, i.e. $\bar{\mathcal{E}} \equiv \sum_j \delta_{\xi_j}$ and define

$$D \equiv \sum_j \delta_{\Lambda_j}, \quad \Lambda_j = \xi_j - \max_i \xi_i. \tag{7.19}$$

D is a point process on $(-\infty.0]$ with an atom at 0.

Corollary 7.6. *Let $x = -at + o(t)$. In the limit $t \to \infty$ the random variables D_t and $x + \max \bar{\mathcal{E}}_t$ are conditionally independent on the event $\{x + \max \bar{\mathcal{E}}_t > b\}$ for any $b \in \mathbb{R}$. More precisely, for any bounded function f, h and $\bar{\phi} \in C_c(\mathbb{R})$,*

$$\begin{aligned} &\lim_{t\to\infty} E\left[f\left(\int \bar{\phi}(z)D_t(dz)\right) h(x + \max \bar{\mathcal{E}}) \big| x + \max \bar{\mathcal{E}} > b\right] \\ =\ & E\left[f\left(\int \bar{\phi}(z)D(dz)\right)\right] \frac{\int_b^\infty h(z)(\sqrt{2}+a)e^{-(\sqrt{2}+a)z}dz}{e^{-(\sqrt{2}+a)b}}. \end{aligned} \tag{7.20}$$

Proof. The proof is essentially identical to the proof of Corollary 4.12 in [3]. Let us outline, for the benefit of the readers, the structure of the proof. First, by Proposition 7.5 the pair $(\bar{\mathcal{E}}_t, \max \bar{\mathcal{E}}_t - x)$, converge under the law conditioned on $\max \bar{\mathcal{E}}_t - x > 0$ to $(\mathcal{E}, e)$, where e is an exponential random variable with parameter $\sqrt{2} + a$ and $\mathcal{E}$ is independent of the precise value of the conditioning. A general continuity lemma, stated and proven as Lemma 4.13 in [3], shows that this implies the convergence of the processes $(D_t, \max \bar{\mathcal{E}}_t - x)$ to a pair $(\mathcal{D}, e)$ where D_t is given in (7.18) is related to $\bar{\mathcal{E}}_t$ by a random shift of its atoms. The fact that $\mathcal{D}$ and e are independent follows from an explicit computation, just as in the proof of Corollary 4.12 in [3]. We do not repeat the details. □

Finally we come to the description of the extremal process as seen from the Poisson process of cluster extremes, which is the formulation of Theorem 1.1.

Theorem 7.7. *Let P_Y be as in (7.8) and let $\{D^{(i)}, i \in \mathbb{N}\}$ be a family of independent copies of the gap-process (7.19) with atoms $\Lambda_j^{(i)}$. Then the point process $\mathcal{E}_t$ converges in law as $t \to \infty$ to a Poisson cluster point process $\mathcal{E}$ given by*

$$\mathcal{E} \stackrel{law}{=} \sum_{i,j} \delta_{p_i + \sigma_2 \Lambda_j^{(i)}}. \tag{7.21}$$

Proof. Also this proof is now very close to that of Theorem 2.1 in [3]. First note that the Laplace functional of the process $\mathcal{E}$ is given by

$$\begin{aligned} &E\left[\exp\left(-\int \phi(x)\mathcal{E}(dx)\right)\right] \\ &= E\left[\exp\left(-\sigma_2 C(a) Y \int_{\mathbb{R}} E\left[1 - e^{-\int \phi(y+x)D(dx)}\right] \sqrt{2}e^{-\sqrt{2}y}dy\right)\right]. \end{aligned} \tag{7.22}$$

Thus, by Theorem 7.2, we have to show that the Laplace functional of the processes Π_t converge to this expression. In the proof of that theorem, we have seen that

$$\begin{aligned} &\lim_{t\uparrow\infty} E\left[\exp\left(-\int \phi(x)\Pi_t(dx)\right)\right] \\ &= E\left[\exp\left(-\sigma_2 Y \lim_{t\uparrow\infty} \int_{-\infty}^0 E\left[1 - \exp\left(-\int \bar{\phi}(z+x)\bar{\mathcal{E}}_t(dx)\right)\right] \frac{e^{-(\sqrt{2}+a)z - a^2t/2}}{\sqrt{2\pi}} dz\right)\right]. \end{aligned} \tag{7.23}$$

We rewrite

$$\begin{aligned} & \int_{-\infty}^{0} E\left[1-\exp\left(-\int \bar\phi(z+x)\bar{\mathcal{E}}_t(dx)\right)\right]\frac{1}{\sqrt{2\pi}}e^{-(\sqrt{2}+a)z-a^2t/2}dz \\ = \quad & \int_{-\infty}^{0} E\left[f\left(\int \left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)\right]\frac{1}{\sqrt{2\pi}}e^{-(\sqrt{2}+a)z-a^2t/2}dz, \end{aligned} \tag{7.24}$$

where $f(x)=1-e^{-x}$, $T_z\bar\phi(x)=\bar\phi(z+x)$, $f(0)=0$. Using the localisation estimate of Proposition 7.4 we have that (7.24) is equal to

$$\Omega_t(B)+\int_{-at-B\sqrt{t}}^{-at+B\sqrt{t}} E\left[f\left(\int \left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)\right]\frac{1}{\sqrt{2\pi}}e^{-(\sqrt{2}+a)z-a^2t/2}dz, \tag{7.25}$$

where $\lim_{B\to\infty}\sup_{t\geq t_0}\Omega_t(B)=0$. Let $m_{\bar\phi}$ be the minimum of the support of $\bar\phi$. we observe that

$$f\left(\int \left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)=0, \tag{7.26}$$

when $z+\max\bar{\mathcal{E}}_t<m_{\bar\phi}$. Moreover, $P\left[z+\max\bar{\mathcal{E}}_t=m_{\bar\phi}\right]=0$. Hence

$$\begin{aligned} & E\left[f\left(\int \left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)\right] \\ = \quad & E\left[f\left(\int \left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)\mathbb{1}_{\{z+\max\bar{\mathcal{E}}_t>m_{\bar\phi}\}}\right] \\ = \quad & E\left[f\left(\int \left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)\big|z+\max\bar{\mathcal{E}}_t>m_{\bar\phi}\right]P\left[z+\max\bar{\mathcal{E}}_t>m_{\bar\phi}\right]. \end{aligned} \tag{7.27}$$

Now by Corollary 7.6, for z in the range of integration in (7.25), on the event we are conditioning on in (7.27), the random variables D_t and $\max\bar{\mathcal{E}}_t+z-m_{\bar\phi}$ converge to independent random variables (D,e), where e is exponential with parameter $\sqrt{2}+a$. Hence

$$\begin{aligned} & \lim_{t\uparrow\infty} E\left[f\left(\int \left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)\big|z+\max\bar{\mathcal{E}}_t>m_{\bar\phi}\right] \\ & =\int_0^\infty(\sqrt{2}+a)e^{-(\sqrt{2}+a)u}E\left[f\left(\int\bar\phi(u+m_{\bar\phi}+x)D(dx)\right)\right]du \\ & =\int_{m_{\bar\phi}}^\infty(\sqrt{2}+a)e^{-(\sqrt{2}+a)(u-m_{\bar\phi})}E\left[f\left(\int\bar\phi(u+x)D(dx)\right)\right]du. \end{aligned} \tag{7.28}$$

Note that this expression is independent of z. Thus it remains to compute the integral of $P\left[z+\max\bar{\mathcal{E}}_t>m_{\bar\phi}\right]$. But this converges to $e^{-(\sqrt{2}+a)m_{\bar\phi}}$ by (6.30) in Lemma 6.4, together with the localisation estimates of Proposition 7.4 (which this time allows to re-extend the range of integration). Putting this together with (7.28) and changing variables $x=\sigma_2 z$ shows that the right-hand side of (7.23) is indeed equal to the right-hand side of (7.22). This proves the theorem. □

8 The case $\sigma_1>\sigma_2$

In this section we proof Theorem 1.3. The existence of the process $\mathcal{E}$ from (1.15) will be a byproduct of the proof.

The following result is contained in the calculation of the maximal displacement in [10].

Lemma 8.1. *([10]) For all $\epsilon > 0, d \in \mathbb{R}$ there exists a constant D large enough such that for t sufficiently large*

$$\mathbb{P}\left[\exists k \le n(t) : x_k(t) > m(t) + d \text{ and } x_k(bt) < m_1(bt) - D\right] < \epsilon. \tag{8.1}$$

Proof of Theorem 1.3. First we establish the existence of a limiting process. Note that $m(t) = m_1(bt) + m_2((1-b)t)$, where $m_i(s) = \sqrt{2}\sigma_i s - \frac{3}{2\sqrt{2}}\sigma_i \log s$. Recall

$$\bar{\phi}(z) = \phi(\sigma_2 z) \tag{8.2}$$

and

$$g_\delta(z) = e^{-\bar{\phi}(-z)} \mathbb{1}_{\{-z \le \delta\}}. \tag{8.3}$$

Using that the maximal displacement is $m(t)$ in this case we can proceed as in the proof of Theorem 6.1 up to (6.9) and only have to control

$$\Psi_t^{<\delta}(\phi) = \mathbb{E}\left[\prod_{i \le n(tb)} \mathbb{E}\left[\prod_{j \le n_i((1-b)t)} g_\delta((m(t) - x_i(bt))/\sigma_2 - \bar{x}_j^i((1-b)t)) \big| \mathcal{F}_{tb}\right]\right], \tag{8.4}$$

where $\bar{x}_j^i((1-b)t)$ are the particles of a standard BBM at time $(1-b)t$ and $x_i(bt)$ are the particles of a BBM with variance σ_1 at time bt. Using Lemma 8.1 and Theorem 1.2 of [10] as in the proof of Theorem 5.1 above, we obtain that (8.4), for t sufficiently large, equals

$$E\left[\prod_{\substack{i \le n(bt)\\ x_i(bt) > m_1(bt) - D}} \mathbb{E}\left[\prod_{j \le n_i((1-b)t)} g_\delta(\tfrac{(m(t) - x_i(bt))}{\sigma_2} - \bar{x}_j^i((1-b)t)) \big| \mathcal{F}_{tb}\right]\right] + O(\epsilon). \tag{8.5}$$

The rest of the proof has an iterated structure. In a first step we show that conditioned on $\mathcal{F}_{bt}$ for each $i \le n(bt)$ the points $\{x_i(bt) + x_j^i((1-b)t) - m(t) | x_i(bt) > m_1(bt) - D\}$ converge to the corresponding points of the point process $x_i(bt) - m_1(bt) + \sigma_2 \tilde{\mathcal{E}}^{(i)}$, where $\tilde{\mathcal{E}}^{(i)}$ are independent copies of the extremal process (1.6) of standard BBM. To this end observe that

$$u_\delta((1-b)t, z) = 1 - \mathbb{E}\left[\prod_{j \le n((1-b)t)} g_\delta(z - \bar{x}_j^i((1-b)t))\right] \tag{8.6}$$

solves the F-KPP equation (3.2) with initial condition $u_\delta(0, z) = 1 - e^{-\bar{\phi}(-z)} \mathbb{1}_{\{-z \le \delta\}}$. Moreover, the assumptions of Lemma 4.9 in [3] are satisfied. Hence (8.5) is equal to

$$\epsilon + \mathbb{E}\left[\prod_{\substack{i \le n(bt)\\ x_i(bt) > m_1(bt) - D}} \left(\mathbb{E}\left[e^{-C(\bar{\phi},\delta) Z e^{-\sqrt{2}\frac{m_1(bt) - x_i(bt)}{\sigma_2}}} \big| \mathcal{F}_{bt}\right](1 + o(1))\right)\right]. \tag{8.7}$$

Here $C(\bar{\phi}, \delta)$ is from standard BBM, i.e.

$$C(\bar{\phi}, \delta) = \lim_{t \uparrow \infty} \sqrt{\tfrac{2}{\pi}} \int_0^\infty u_\delta(t, y + \sqrt{2}t) y e^{\sqrt{2}y} dy, \tag{8.8}$$

see Eq. 4.49 in [3]. Note furthermore that already in (8.7) the concatenated structure of the limiting point process becomes visible. In a second step we establish that the points $x_i(bt) - m_1(t)$ that have a descendant in the lead at time t converge to $\widetilde{\mathcal{E}}$.

Define

$$h_{\delta,D}(y) \equiv \begin{cases} \mathbb{E}\left[\exp\left(-C(\bar{\phi},\delta)Ze^{-\sqrt{2}\frac{\sigma_1}{\sigma_2}y}\right)\right], & \text{if } \sigma_1 y < D, \\ 1, & \text{if } \sigma_1 y \geq D. \end{cases} \tag{8.9}$$

Then the expectation in (8.7) can be written as (we ignore the error term $o(1)$ which is easily controlled using that the probability that the number of terms in the product is larger than N tends to zero as $N \uparrow \infty$, uniformly in t)

$$\mathbb{E}\left[\prod_{i \leq n(bt)} h_{\delta,D}(m_1(bt)/\sigma_1 - \bar{x}_i(t))\right], \tag{8.10}$$

where now $\bar{x}$ is standard BBM. Defining

$$v_{\delta,D}(t,z) = 1 - \mathbb{E}\left[\prod_{i \leq n(t)} h_{\delta,D}(z - \bar{x}_i(bt))\right], \tag{8.11}$$

$v_{\delta,D}$ is a solution of the F-KPP equation (3.2) with initial condition $v_{\delta,D}(0,z) = 1 - h_{\delta,D}(z)$. But this initial condition satisfies the assumptions of Bramson's Theorem A in [6] and therefore,

$$v_{\delta,D}(t, m(t) + x) \to \mathbb{E}\left[e^{-\tilde{C}(D,Z,C(\bar{\phi},\delta))\tilde{Z}e^{-\sqrt{2}x}}\right]. \tag{8.12}$$

where $\tilde{Z}$ is an independent copy of Z and

$$\tilde{C}(D,Z,C(\bar{\phi},\delta)) = \lim_{t\uparrow\infty} \sqrt{\tfrac{2}{\pi}} \int_0^\infty v_{\delta,D}(t, y + \sqrt{2}t) y e^{\sqrt{2}y} dy. \tag{8.13}$$

By the same argumentation as in standard BBM setting (see [3]) one obtains that

$$\tilde{C}(Z,C(\bar{\phi},\delta)) \equiv \lim_{D\uparrow\infty} \tilde{C}(D,Z,C(\bar{\phi},\delta)) = \lim_{t\uparrow\infty} \sqrt{\tfrac{2}{\pi}} \int_0^\infty v_\delta(t, y + \sqrt{2}t) y e^{\sqrt{2}y} dy, \tag{8.14}$$

where v_δ is the solution of the F-KPP equation with initial condition $v(0,z) = 1 - h_\delta(z)$ with

$$h_\delta(z) = \mathbb{E}\left[\exp\left(-C(\bar{\phi},\delta)Ze^{-\sqrt{2}\frac{\sigma_1}{\sigma_2}z}\right)\right]. \tag{8.15}$$

The next step is to take the limit $\delta \to \infty$. Using Lemma 4.10 of [3] we have that $C(\bar{\phi},\delta)$ is monotone decreasing in δ and $\lim_{\delta\to\infty} C(\bar{\phi},\delta) = C(\bar{\phi})$, exists and is strictly positive, where

$$C(\bar{\phi}) = \lim_{t\uparrow\infty} \sqrt{\tfrac{2}{\pi}} \int_0^\infty u(t, y + \sqrt{2}t) y e^{\sqrt{2}y} dy. \tag{8.16}$$

Here $u(t,x)$ is a solution to the F-KPP equation (3.2) with initial condition $u(0,x) = 1 - e^{-\bar{\phi}(-x)}$. Using the same monotonicity arguments shows that also

$$\lim_{\delta\to\infty} \tilde{C}(Z, C(\bar{\phi},\delta)) = \tilde{C}(Z, C(\bar{\phi})). \tag{8.17}$$

Therefore, taking the limit first as $D \uparrow \infty$ and then $\delta \uparrow \infty$ in the left-hand side of (8.12), we get that

$$\begin{aligned} \lim_{t\to\infty} \Psi_t(\phi(\cdot + x)) &= \lim_{\delta\uparrow\infty} \lim_{t\to\infty} \Psi_t^{<\delta}(\phi(\cdot + x)) \qquad (8.18) \\ &= \lim_{\delta\uparrow\infty} \lim_{D\uparrow\infty} \lim_{t\to\infty} v_{\delta,D}(t, m(t) + x) = \mathbb{E}\left[e^{-\tilde{C}(Z,C(\bar{\phi}))\tilde{Z}e^{-\sqrt{2}x}}\right]. \end{aligned}$$

To see that the constants $\tilde{C}(Z, C(\bar{\phi}))$ are strictly positive, one uses the Laplace functionals $\Psi_t(\phi)$ are bounded from above by

$$\mathbb{E}\left[\exp\left(-\phi\left(\max_{i\le n(bt)} x_i(bt) + \max_{j\le n_1((1-b)t)} x_j^1((1-b)t) - m(t)\right)\right)\right] \tag{8.19}$$

Here we used that the offspring of any of the particles at time bt has the same law. So the sum of the two maxima in the expression above has the same distribution as the largest descendent at time t off the largest particle at time bt. The limit of Eq. (8.19) as $t \uparrow \infty$ exists and is strictly smaller than 1 by the convergence in law of the recentered maximum of a standard BBM. But this implies the positivity of the constants $\tilde{C}$. Hence a limiting point process exists. Finally, one may easily check that the right hand side of (8.18) coincides with the Laplace functional of the point process defined in (1.15) by basically repeating the computations above. □

Remark 8.2. *Note that in particular, the structure of the variance profile is contained in the constant $\tilde{C}(D, Z, C(\bar{\phi}, \delta))$ and that also the information on the structure of the limiting point process is contained in this constant. In fact, we see that in all cases we have considered in this paper, the Laplace functional of the limiting process has the form*

$$\lim_{t\uparrow\infty} \Psi_t(\phi(\cdot + x)) = \mathbb{E}\exp\left(-C(\phi)Me^{-\sqrt{2}x}\right), \tag{8.20}$$

where M is a martingale limit (either Y of Z) and C is a map from the space of positive continuous functions with compact support to the real numbers. This function contains all the information on the specific limiting process. This is compatible with the finding in [16] in the case where the speed is a concave function of s/t. The universal form (8.20) is thus misleading and without knowledge of the specific form of $C(\phi)$, (8.20) contains almost no information.

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